CANONICAL HAMMING GEOMETRY FROM THE SINGULAR 3-ADIC HECKE FACTOR AT LEVEL 163

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Abstract. Let $V = S_2(\Gamma_0(163))$ and let $Q = \ker(T_2^2 : V_{\mathbf{F}_3} \to V_{\mathbf{F}_3})$ be the canonical 3-dimensional mod-3 Hecke summand. The projective plane $\mathbf{P}(Q)$ has 13 points. We show that the natural incidence structure on $\mathbf{P}(Q)$ is a 2-$(13, 4, 1)$ design, and that the unique such design is the projective plane $\mathrm{PG}(2, \mathbf{F}_3)$. This identifies the parity-check geometry of the $[13, 10, 3]_3$ ternary Hamming code as a canonical invariant of the Hecke module at level 163.

The construction is canonical: it depends on no choice of basis, no primitive root, and no coordinates. Every step is either functorial or forced by a uniqueness theorem. The Heegner syndrome algebra $A_H \cong \mathbf{F}_3 \times \mathbf{F}_3[\varepsilon]/(\varepsilon^2)$ acts on $\mathbf{P}(Q)$ through its projective unit group C_6, with orbit decomposition $1 + 1 + 2 + 3 + 6$.

We also report a census of singular 3-adic Hecke factors at all prime levels $N \leq 1000$, finding seventeen split-nodal singularities and thirteen non-split-nodal singularities. Among the class-number-one primes, 163 is the only level with a split node.

Contents

1. Introduction

The ternary Hamming code $\mathcal{H}_3 = [13, 10, 3]_3$ is the unique perfect single-error-correcting code over $\mathbf{F}_3$ of length 13. Its parity-check matrix has 3 rows and 13 columns, one for each point of the projective plane $\mathrm{PG}(2, \mathbf{F}_3)$. The code, the plane, and the 2-$(13, 4, 1)$ block design are three faces of the same combinatorial object.

In this note we show that this object arises canonically from the Hecke algebra at level $N = 163$, the largest class-number-one Heegner prime.

The construction is:

(i) Start from $V = S_2(\Gamma_0(163))$, $\dim_{\mathbf{Q}} V = 13$.
(ii) Reduce modulo 3 to obtain $V_{\mathbf{F}_3}$, using the integral modular-symbols lattice.
(iii) Take the x^2-primary component of T_2: $Q = \ker(T_2^2) \subset V_{\mathbf{F}_3}$, $\dim Q = 3$.
(iv) Form the projective plane $\mathbf{P}(Q)$: this has 13 points.
(v) The natural incidence structure on $\mathbf{P}(Q)$ is a 2-$(13, 4, 1)$ design.
(vi) By uniqueness of the projective plane of order 3, this is $\mathrm{PG}(2, \mathbf{F}_3)$.

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(vii) The kernel of the parity-check map $\mathbf{F}_3^{\mathbf{P}(Q)} \to Q$ is the ternary Hamming code $[13, 10, 3]_3$.

No choice is made at any step. Step (iii) is canonical because the x^2-primary component is determined by the Hecke action, not by a basis. Step (vi) is forced by uniqueness: there is exactly one 2-$(13, 4, 1)$ design up to isomorphism [1].

Main results.

(A) The Hecke module at level 163 canonically produces the projective plane $\mathrm{PG}(2, \mathbf{F}_3)$ and thereby the ternary Hamming code (Theorem 3.2).
(B) The Heegner syndrome algebra A_H acts on this plane through $C_6 \subset \mathrm{PGL}(3, \mathbf{F}_3)$, with orbit structure $1 + 1 + 2 + 3 + 6$ (Theorem 4.1).
(C) A census of prime levels $N \leq 1000$ finds seventeen split-nodal and thirteen non-split-nodal 3-adic Hecke singularities (Proposition 6.1). Among the class-number-one primes, 163 is the unique split-nodal level. Among all split-nodal levels ≤ 500, only three (127, 163, 337) have $\dim Q_N = 3$ and carry the $\mathrm{PG}(2, \mathbf{F}_3)$ Hamming geometry (Corollary 6.2).

2. The canonical summand and its projective plane

Let $V_{\mathbf{Z}} \subset S_2(\Gamma_0(163))$ be the integral modular-symbols lattice, and set $V_{\mathbf{F}_3} = V_{\mathbf{Z}}/3V_{\mathbf{Z}}$.

Proposition 2.1. *The operator T_2 on $V_{\mathbf{F}_3}$ has characteristic polynomial*

$$\chi_{T_2}(x) \equiv x^3(x^4 + 2x^3 + 2)(x^6 + x^4 + x^3 + x + 1) \pmod 3.$$

The three factors are pairwise coprime, giving a Hecke-stable decomposition

$$V_{\mathbf{F}_3} = Q \oplus W_4 \oplus W_6,$$

where $Q = \ker(T_2^2)$ has dimension 3, W_4 has dimension 4, and W_6 has dimension 6.

Proof. Direct computation in SageMath; coprimality is verified by $\gcd(x^2, x^4+2x^3+2) = 1$ and similarly for the other pairs in $\mathbf{F}_3[x]$. □

Definition 2.2. The *Hecke projective plane* at level 163 is

$$\mathbf{P}(Q) = \{\text{one-dimensional } \mathbf{F}_3\text{-subspaces of } Q\}.$$

Since $\dim Q = 3$, the set $\mathbf{P}(Q)$ has $(3^3 - 1)/(3 - 1) = 13$ points.

3. The canonical Hamming geometry

Definition 3.1. A *line* in $\mathbf{P}(Q)$ is the projectivization of a 2-dimensional subspace of Q.

Theorem 3.2 (Canonical Hamming geometry)**.** *The incidence structure $(\mathbf{P}(Q), \mathcal{L})$, where $\mathcal{L}$ is the set of lines, satisfies:*

(a) there are 13 points and 13 lines;
(b) each line contains exactly 4 points;
(c) each point lies on exactly 4 lines;
(d) any two distinct points determine exactly one line.

This is a 2-$(13, 4, 1)$ balanced incomplete block design. By the uniqueness theorem for projective planes of order 3 [1],

$$(\mathbf{P}(Q), \mathcal{L}) \cong \mathrm{PG}(2, \mathbf{F}_3).$$

Proof. Properties (a)–(d) are the standard properties of $\mathbf{P}^2$ over any field with 3 elements. Since Q is a 3-dimensional $\mathbf{F}_3$-vector space, $\mathbf{P}(Q)$ is abstractly isomorphic to $\mathbf{P}^2(\mathbf{F}_3)$, and the incidence of points and lines is that of the standard projective plane. The 2-$(13, 4, 1)$ design is unique up to isomorphism [1], so the isomorphism class of the incidence structure is independent of any basis choice. □

Corollary 3.3 (The Hamming code)**.** *Let* $E = \mathbf{F}_3^{\mathbf{P}(Q)}$ *be the free module on the* 13 *points. The parity-check map*

$$H_Q : E \to Q$$

sending each basis vector to a nonzero representative of the corresponding projective point has rank $H_Q = 3$*, and*

$$\mathcal{C}_Q := \ker(H_Q) \subset E$$

is a $[13, 10, 3]_3$ *ternary code. This is the ternary Hamming code, determined up to* $\mathrm{GL}(Q)$*-equivalence (i.e., column permutation and scaling).*

Proof. After choosing a basis of Q, the 13 projective points give 13 distinct nonzero column vectors in $\mathbf{F}_3^3$, forming a standard Hamming parity-check matrix [2]. The resulting code has parameters $[13, 10, 3]_3$. Changing the basis of Q acts by $\mathrm{GL}(3, \mathbf{F}_3)$ on all columns simultaneously and therefore preserves the code up to equivalence. □

Remark 3.4. The canonicity chain is:

$$S_2(\Gamma_0(163)) \xrightarrow{\text{mod } 3} V_{\mathbf{F}_3} \xrightarrow{\ker(T_2^2)} Q \xrightarrow{\mathbf{P}} \mathbf{P}(Q) \xrightarrow{\text{incidence}} \mathrm{PG}(2, \mathbf{F}_3) \xrightarrow{\ker(H_Q)} [13, 10, 3]_3.$$

Each arrow is either functorial (mod-3 reduction, primary decomposition, projectivization, parity-check kernel) or forced by a uniqueness theorem (the 2-$(13, 4, 1)$ design is unique).

4. The Hecke symmetry of the plane

The seven Heegner operators T_d for $d \in \{3, 7, 11, 19, 43, 67, 163\}$ act on Q through the syndrome algebra

$$A_H = \mathrm{Span}_{\mathbf{F}_3}\{I, E, N\} \cong \mathbf{F}_3 \times \mathbf{F}_3[\varepsilon]/(\varepsilon^2),$$

with $E^2 = E$, $N^2 = EN = NE = 0$ (see [3]). The invertible elements of A_H act on $\mathbf{P}(Q)$ by projectivization.

Theorem 4.1 (Hecke symmetry of the Hamming plane)**.** *The projective unit group* $A_H^\times/\mathbf{F}_3^\times \cong C_6$ *acts on* $\mathbf{P}(Q)$ *with orbit decomposition*

$$1 + 1 + 2 + 3 + 6 = 13.$$

The cycle types of the invertible Heegner operators are:

Operator	*Projective order*	*Cycle type on* 13 *points*
T_7	3	3+3+3+1+1+1+1
T_{43}	6	6+3+2+1+1
T_{67}	6	6+3+2+1+1
T_{163}	2	2+2+2+2+1+1+1+1+1

These four operators generate the full projective unit group. The two fixed points are the eigenspaces of the idempotent $E = T_3|_Q$.

Proof. The unit group computation follows from the algebra structure: $|A_H^\times| = |\mathbf{F}_3^\times| \cdot |(\mathbf{F}_3[\varepsilon]/(\varepsilon^2))^\times| = 2 \cdot 6 = 12$, so $|A_H^\times/\mathbf{F}_3^\times| = 6$. The cycle types and orbit sizes are computed by explicit matrix action on the 13 points of $\mathbf{P}(Q)$, using the restricted Hecke matrices from the appendix of [4]. The two fixed points correspond to the E-eigenspace $\langle e_1 \rangle$ (where E acts as 1) and the kernel of E restricted to the pair block. □

Remark 4.2. The group C_6 embeds in $\mathrm{PGL}(3, \mathbf{F}_3)$, which is the full automorphism group of $\mathrm{PG}(2, \mathbf{F}_3)$. Since $|\mathrm{PGL}(3, \mathbf{F}_3)| = 11{,}232$, the Hecke symmetry is a small but arithmetically distinguished subgroup. The orbit decomposition $1 + 1 + 2 + 3 + 6$ is a partition of the 13 code positions into Hecke-equivalent classes.

5. The syndrome algebra as decoder

The classical role of the syndrome in coding theory is error detection: given a received word $r \in \mathbf{F}_3^{13}$, the syndrome is $s = H_Q(r) \in Q$. If $s = 0$, no error is detected. If $s \neq 0$, the syndrome identifies the error position as the projective point $[s] \in \mathbf{P}(Q)$.

Proposition 5.1. *The Heegner syndrome algebra A_H acts on the syndrome space Q. In particular:*

(a) *The idempotent $E = T_3|_Q$ splits the syndrome into a semisimple component (the E-eigenspace) and a pair component (the* $\ker(E)$*-subspace).*
(b) *The nilpotent $N = T_{11}|_Q$ acts within the pair component. It maps one syndrome to a neighboring syndrome within the nilpotent fiber, without changing the semisimple part.*
(c) *The collision $T_{43}|_Q = T_{67}|_Q$ means that the syndromes of these two Heegner operators are indistinguishable at the residue level. They separate only after* 3*-adic lifting.*

Proof. Direct from the algebra relations $E^2 = E$, $N^2 = EN = 0$, and the explicit operator images in Theorem 4.1 of [3]. □

Remark 5.2. The name "syndrome algebra" is thus doubly justified: it is both the residue algebra of the singular 3-adic Hecke node [4] and the algebra that acts on the syndrome space of the canonically associated Hamming code. The two meanings coincide because the 3-dimensional space Q plays both roles simultaneously: it is the mod-3 Hecke summand and the syndrome space of the code.

6. Census of singular 3-adic Hecke factors

We apply the same method — mod-3 primary decomposition of T_2, identification of 3-dimensional summands, and 3-adic lifting — to all prime levels $N \leq 500$.

Proposition 6.1 (Census). *Among the* 91 *prime levels $N \leq 500$ (respectively* 164 *prime levels $N \leq 1000$):*

(a) 56 *(resp.* 95*) levels have a smooth* 3*-adic Hecke algebra;*
(b) 20 *(resp.* 39*) levels have a cuspidal singularity;*
(c) 8 *(resp.* 17*) levels have a split-nodal singularity;*
(d) 7 *(resp.* 13*) levels have a non-split-nodal singularity.*

The split-nodal levels up to 1000 *and their syndrome dimensions are:*

N	*127*	*163*	*199*	*307*	*337*	*449*	*487*	*499*	$\cdots$
$\dim Q_N$	*3*	*3*	*4*	*2*	*3*	*5*	*7*	*6*	

Continuing to 1000*: $N \in \{541, 577, 631, 773, 811, 829, 857, 883, 919\}$.*

The non-split-nodal levels up to 1000 *are: $N \in \{71, 269, 271, 359, 421, 439, 443, 509, 523, 571, 601, 947, 997\}$.*

Proof. Systematic computation in SageMath. For each prime N, we compute T_2 on $V_{\mathbf{F}_3}$, extract $Q_N = \ker(T_2^2)$, and classify the 3-adic local factor by testing for a mod-3 idempotent and its $\mathbf{Z}_3$-lift. Full data and scripts are available in the accompanying repository. □

Corollary 6.2. *The canonical Hamming geometry* $\mathrm{PG}(2, \mathbf{F}_3)$ *arises exactly when* $\dim Q_N = 3$*. Among the split-nodal levels ≤ 500, this occurs at $N \in \{127, 163, 337\}$. Of these, only $N = 163$ is a class-number-one Heegner prime.*

Corollary 6.3. *Among the four class-number-one prime levels $N \in \{19, 43, 67, 163\}$, the dimensions of Q_N are*

$$1, \quad 0, \quad 0, \quad 3,$$

respectively. Level 163 *is the unique class-number-one prime with a split-nodal singularity, a* 3*-dimensional syndrome summand, and a canonical* $\mathrm{PG}(2, \mathbf{F}_3)$ *Hamming geometry.*

Remark 6.4. Higher-dimensional syndrome summands ($\dim Q_N > 3$) produce higher-order projective geometries: $\mathbf{P}(Q_{199})$ has 40 points ($\dim Q_{199} = 4$), $\mathbf{P}(Q_{449})$ has 121 points ($\dim Q_{449} = 5$), and so on. The Hamming codes at these levels have longer block lengths and different parameters. The $[13, 10, 3]_3$ ternary Hamming code is specific to $\dim Q_N = 3$.

Remark 6.5. The classification problem — characterize which prime levels N produce singular 3-adic Hecke factors, determine their local type and syndrome dimension — is a natural next step. The data suggest that split-nodal singularities become more frequent and higher-dimensional as N grows. We leave the asymptotic analysis to future work.

7. Summary

The singular 3-adic Hecke factor at level 163 produces a canonical chain of objects:

$$\begin{aligned} \text{Hecke algebra}_{163} &\longrightarrow \text{mod-3 summand } Q \\ &\longrightarrow \text{projective plane } \mathbf{P}(Q) \cong \mathrm{PG}(2, \mathbf{F}_3) \\ &\longrightarrow \text{Hamming code } [13, 10, 3]_3. \end{aligned}$$

The syndrome algebra $A_H \cong \mathbf{F}_3 \times \mathbf{F}_3[\varepsilon]/(\varepsilon^2)$ is simultaneously:

(i) the residue shadow of the 3-adic node $\mathbf{Z}_3[\eta]/(\eta^2 - 3\eta)$;
(ii) the algebra acting on the syndrome space of the Hamming code;
(iii) the source of a C_6 symmetry of the Hamming plane with orbits $1 + 1 + 2 + 3 + 6$.

Among all prime levels ≤ 1000, seventeen exhibit split-nodal 3-adic singularities. Of these, only three (127, 163, 337) have $\dim Q_N = 3$ and therefore carry the $\mathrm{PG}(2, \mathbf{F}_3)$ Hamming geometry. Of those three, only 163 is a class-number-one Heegner prime with the full seven-operator syndrome algebra.

The Hamming code was not constructed here. It was found.

Computational verification. All results were computed in SageMath. The census data and verification scripts are available in the accompanying repository.

References

[1] M. Hall Jr., *Uniqueness of the projective plane with* 57 *points*, Proc. Amer. Math. Soc. **4** (1953), 912–916. *Note:* The uniqueness of the projective plane of order 3 (13 points) is classical; see also R. H. Bruck and H. J. Ryser, *The nonexistence of certain finite projective planes*, Canad. J. Math. **1** (1949), 88–93.
[2] J. H. van Lint, *Introduction to Coding Theory*, third ed., Springer, 1999.
[3] R. A. Hoekstra, *The Heegner syndrome algebra on the canonical mod-*3 *Hecke quotient at level* 163, preprint, 2026.
[4] R. A. Hoekstra, *The singular* 3*-adic Hecke node at level* 163*: presentation, ideal theory, and module classification*, preprint, 2026.
[5] The Sage Developers, *SageMath, the Sage Mathematics Software System*, `https://www.sagemath.org`.

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