A CENSUS OF SINGULAR 3-ADIC HECKE FACTORS AT PRIME LEVEL

R. S. HOEKSTRA

Abstract. We define a local singularity type for prime level N in terms of the mod-3 Hecke action on the cuspidal modular-symbols space. Using SageMath, we scan all prime levels $11 \le N \le 1000$ and classify each level into exactly one of:

$$\{\text{smooth, cusp, split node, non-split node}\}.$$

The output is a full census and an explicit data table that supports an asymptotic conjecture for split-node density.

1. Setup and Classification Rule

Let N be a prime and let

$$V_N = S_2(\Gamma_0(N)) \otimes \mathbf{Q}, \qquad M_N = V_N \otimes_{\mathbf{Q}} \mathbf{F}_3.$$

Let T_2 be the Hecke operator T_2 and let

$$Q_N := \ker\left(T_2^2 \colon M_N \to M_N\right), \qquad q_N = \dim_{\mathbf{F}_3} Q_N, \qquad f_N = \text{minimal polynomial of } T_2|_{Q_N} \text{ over } \mathbf{F}_3.$$

For $q_N > 0$ we define the *syndrome algebra*

$$\mathcal{A}_N := \operatorname{span}_{\mathbf{F}_3}\{T_\ell|_{Q_N} \mid \ell \in \{3,5,7,11,13,17,19,23\},\ \ell \ne N\} \subset \operatorname{End}_{\mathbf{F}_3}(Q_N),$$

as in the computation script below.

Definition 1 (Category rule)**.** *We assign one of four categories to level N as follows:*

- ***smooth*** *iff $q_N = 0$;*
- ***split node*** *iff $q_N > 0$ and $\mathcal{A}_N$ contains a nontrivial idempotent that lifts to $\mathbf{Z}_3$;*
- ***cusp*** *iff $q_N > 0$, $f_N = x$, and no split idempotent lift exists;*
- ***non-split node*** *iff $q_N > 0$, $f_N = x^2$, and no split idempotent lift exists.*

The split/non-split distinction is implemented by the same Hensel lifting test used in `src/hecke/hecke_prime`

2. Computation

The classification is computed by

- computing a basis of M_N from `ModularSymbols(N,2).cuspidal_subspace()`,
- forming Q_N,
- computing $\mathcal{A}_N$ on Q_N from a small prime Hecke set,
- searching for a nontrivial mod-3 idempotent and lifting it to $\mathbf{Z}_3$.

Theorem 2 (Numerical Census: prime levels $11 \le N \le 500$)**.** *The scan returns exactly*

$$\#\textit{smooth} = 56, \quad \#\textit{cusp} = 20, \quad \#\textit{split node} = 8, \quad \#\textit{non-split node} = 7.$$

Smooth levels are the complement of the 44 non-smooth levels.

Theorem 3 (Numerical Census: prime levels $11 \le N \le 1000$)**.** *The scan over the 164 prime levels in this range gives*

$$\#\textit{smooth} = 95, \quad \#\textit{cusp} = 39, \quad \#\textit{split node} = 17, \quad \#\textit{non-split node} = 13.$$

Date: March 20, 2026.

3. Observed Split Nodes

The split-node primes in the range $N \leq 1000$ are

$$127, 163, 199, 307, 337, 449, 487, 499, 541, 577, 631, 773, 811, 829, 857, 883, 919.$$

The non-split-node primes in this range are

$$71, 269, 271, 359, 421, 439, 443, 509, 523, 571, 601, 947, 997.$$

The cusp levels are

$$19, 37, 73, 101, 109, 113, 131, 149, 151, 167, 181, 223, 293, 379, 383, 397, 431, 433, 463, 467, 503, 521, 557, 587, 593, 59$$

4. qdim-2+ Block

Across $11 \leq N \leq 1000$, exactly 35 levels have $q_N \geq 2$. The values of $\dim_{\mathbf{F}_3} Q_N$ occurring in each category are:

$$\text{cusp: } 1, 2; \quad \text{split node: } 2, 3, 4, 5, 6, 7; \quad \text{non-split node: } 2, 4; \quad \text{smooth: } 0.$$

5. Asymptotics

The split-node counts up to these points are

$$N \leq 500: \ 8, \qquad N \leq 1000: \ 17.$$

At these values this is close to linear growth in the number of primes scanned:

$$\frac{8}{91} \approx 0.088, \qquad \frac{17}{164} \approx 0.104.$$

This motivates the conjecture that split-node primes have positive density among prime levels (with respect to prime counting in this family), and that non-split nodes also persist with positive lower density.

Conjecture 4. *The quantities*

$$\frac{\#\{\text{split node } N \leq X\}}{\#\{p \leq X : \ p \text{ prime}, \ p \neq 3\}} \quad \text{and} \quad \frac{\#\{\text{non-split node } N \leq X\}}{\#\{p \leq X : \ p \text{ prime}, \ p \neq 3\}}$$

have nonzero limits as $X \to \infty$.

6. Reproducibility

All classification data are generated by `src/hecke/hecke_prime_node_census.py`, with outputs `src/hecke/hecke_prime_node_census.json`, `src/hecke/hecke_prime_node_census.md`, `src/hecke/hecke_prime_node_census_upto_1000.json`, and `src/hecke/hecke_prime_node_census_upto_100` Exact matrices and syndrome-algebra consistency checks are in `src/hecke/hecke_syndrome_lift.py`, `src/heegner/heegner_syndrome_algebra.py`.

References

[1] The Sage Developers, *SageMath*, `https://www.sagemath.org`.

Unaffiliated