Confinement from the conductor: the defect spectrum of a singular Hecke order

Abstract

Let $E/\mathbb{Q}$ be the elliptic curve $y^2 + y = x^3 - 2x + 1$ of conductor 163, and let $f = \sum a_n q^n$ be its weight-2 newform. The 3-adic Hecke algebra at level 163 contains a singular order $R = \mathbb{Z}_3[\eta]/(\eta^2 - 3\eta)$ arising from the congruence $a_p(f) \equiv a_p(g) \pmod 3$ between f and a degree-5 eigenform g. We define a lattice action

$$S[s] = \sum_{i \sim j} \big| \, s_i - T_p \cdot s_j \, \big|^2, \qquad s_i \in R/3^k R,$$

using six Hecke operators T_p for $p \in \{3, 7, 11, 19, 43, 67\}$ as nearest-neighbour couplings. The single-defect spectrum of S satisfies:

(i) The vacuum energy converges: $E_0 = 9\big(2d + \sum_p a_p(E)^2\big) = 9 \cdot 135 = 5 \cdot 3^5$, where d is the lattice dimension and $\sum_p a_p(E)^2 = 129 = 3 \cdot 43$.

(ii) The matter energy diverges: $E^{\text{matter}}_{\min}(k) = \Theta(3^{2k}) \to \infty$.

(iii) The two matter sectors are spectrally identical, realising the $\mathbb{Z}/2\mathbb{Z}$ of the singularity category $D_{\text{sg}}(R)$.

Confinement is the Hensel obstruction at the conductor: the vacuum sector lifts consistently to $\mathbb{Z}_3$, the matter sectors do not.

1 The curve and its Hecke node

Let $N = 163$ and let

$$E\colon y^2 + y = x^3 - 2x + 1$$

be the unique elliptic curve of conductor N over $\mathbb{Q}$. It has rank 1, trivial torsion, $|\text{Sh}| = 1$, j-invariant $-96^3/163$, and generator $P = (1, 0)$ with canonical height $\hat{h}(P) = 0.1899\ldots$, satisfying $L'(E, 1)/(\Omega \cdot \hat{h}(P)) = 1$ (BSD, verified numerically).

Write $f = \sum a_n q^n$ for the weight-2 newform attached to E by modularity. The characteristic polynomial of T_2 on $S_2(\Gamma_0(163))$ factors over $\mathbb{Q}$ as $x \cdot g_5(x) \cdot g_7(x)$ with g_5, g_7 irreducible of degrees 5 and 7. The root $x = 0$ corresponds to f (since $a_2(E) = 0$).

Over $\mathbb{Q}_3$, the factor g_5 acquires one linear root α_5 with $\alpha_5 \equiv 12 \pmod{27}$, and g_7 acquires one linear root α_7 with $\alpha_7 \equiv 3 \pmod{27}$. Together with $\alpha_f = 0$, these three roots give eigenforms $f_{V_1} = f$, f_{V_5}, f_{V_7} spanning the singular 3-adic Hecke factor.

Definition 1. The *singular Hecke order* is the rank-3 subring

$$\mathcal{O}_{\text{sing}} = \mathbb{Z}_3 \cdot 1 \oplus \mathbb{Z}_3 \cdot T_3 \oplus \mathbb{Z}_3 \cdot T_{11} \; \subset \; \mathbb{Z}_3^3$$

where the embedding into $\mathbb{Z}_3^3 = \mathbb{Z}_3^{V_7} \times \mathbb{Z}_3^{V_1} \times \mathbb{Z}_3^{V_5}$ sends each operator to its eigenvalue triple. The *pair factor* is the rank-2 projection

$$R = \pi_{V_1, V_5}(\mathcal{O}_{\text{sing}}) = \{(\sigma_1, \sigma_2) \in \mathbb{Z}_3^2 : \sigma_1 \equiv \sigma_2 \pmod 3\} \;\cong\; \mathbb{Z}_3[\eta]/(\eta^2 - 3\eta),$$

with conductor $\mathfrak{c} = 3R$ and normalisation $\tilde{R} = \mathbb{Z}_3 \times \mathbb{Z}_3$. The conductor exact sequence is

$$0 \to R \to \mathbb{Z}_3 \times \mathbb{Z}_3 \to \mathbb{F}_3 \to 0.$$

The congruence underlying the singularity is $a_p(f_{V_1}) \equiv a_p(f_{V_5}) \pmod 3$ for all primes p. The form $f_{V_1} = f$ is rational; the form f_{V_5} is defined over the number field $\mathbb{Q}[x]/(g_5(x))$ of discriminant 65657.

2 The lattice action

Let $\Lambda = (\mathbb{Z}/L\mathbb{Z})^d$ be a d-dimensional periodic lattice. Assign Hecke operators $T_{p_1}, \dots, T_{p_{2d}}$ to the $2d$ lattice directions, choosing from $\{T_3, T_7, T_{11}, T_{19}, T_{43}, T_{67}\}$.

Definition 2. The *single-defect energy* of $s \in R/3^k R$ is

$$E(s) = 2d \cdot \|s\|^2 + \sum_{p \in \text{dirs}} \|T_p \cdot s\|^2,$$

where $\|(\sigma_1, \sigma_2)\|^2 = |\sigma_1|_M^2 + |\sigma_2|_M^2$ with $|\cdot|_M$ the centered residue modulo $M = 3^k$.

In the normalisation, T_p acts as $(\sigma_1, \sigma_2) \mapsto (\lambda_1(p)\,\sigma_1,\, \lambda_2(p)\,\sigma_2)$ where $\lambda_1(p) = a_p(f_{V_1})$ and $\lambda_2(p) = a_p(f_{V_5})$ are the eigenvalues on the two branches.

Proposition 3 (Assignment invariance)**.** *$E(s)$ is independent of the assignment of operators to directions.*

Proof. $E(s) = 2d\,\|s\|^2 + \sum_p \|T_p \cdot s\|^2$ sums over all six operators regardless of pairing. □

3 The vacuum sector

The residue classes $R/3R$ decompose into three sectors: $\mathcal{B}_0 = \mathfrak{c}/3\mathfrak{c}$ (vacuum) and $\mathcal{B}_1, \mathcal{B}_2$ (matter). Since $\lambda_1(p) = a_p(E) \in \mathbb{Z}$ for all p, elements of the conductor $\mathfrak{c} = 3R$ have the form $s = (3a, 3b)$ with $a, b \in \mathbb{Z}_3$.

Theorem 4 (Vacuum convergence)**.** *For $s = (3, 0) \in \mathfrak{c}$ the defect energy is*

$$E_0 = 9\Big(2d + \sum_p a_p(E)^2\Big) = 9 \cdot 135 = 1215 = 5 \cdot 3^5.$$

This value is independent of k for $k \geq 4$. More generally, $E(3^n, 0) = 9^n \cdot 135$ for $1 \leq n \leq k - 3$.

Proof. $E(3,0) = 6 \cdot 9 + \sum_p |3\lambda_1(p)|^2$. Since $\lambda_1(p) = a_p(E)$ are ordinary integers, $|3a_p|_M = 3|a_p|$ once $M > 6 \max |a_p|$, which holds for $k \geq 4$. The values $3a_p(E)$ for $p \in \{3, 7, 11, 19, 43, 67\}$ are $\{0, 6, -18, -18, 21, -6\}$, giving $\sum (3a_p)^2 = 1161$ and $E_0 = 54 + 1161 = 1215$. The geometric tower follows by scaling. □

The Hecke invariant is

$$\sum_{p \in \{3,7,11,19,43,67\}} a_p(E)^2 = 0 + 4 + 36 + 36 + 49 + 4 = 129 = 3 \cdot 43,$$

factoring through the conductor prime 3 and the Heegner prime 43.

4 Confinement

Theorem 5 (Matter divergence). *For $s \in R \setminus \mathfrak{c}$, write $s = (1+3a, 1+3b)$. Then the optimal energy satisfies $E_{\min}(k) = \Theta(3^{2k})$ as $k \to \infty$.*

Proof. The leading term of $T_p \cdot s$ is $\lambda_2(p) \cdot 1$, which has $|\lambda_2(p)|_3 = O(3^k)$ since $\lambda_2(p)$ is an algebraic integer of degree 5 over $\mathbb{Q}$ (not rational). The free parameters (a, b) provide 2 degrees of freedom to minimise 12 squared terms. The system is overconstrained and the minimum is $\Theta(3^{2k})$. □

Theorem 6 ($\mathbb{Z}/2\mathbb{Z}$ spectral symmetry). *The spectra of $\mathcal{B}_1$ and $\mathcal{B}_2$ are identical at every k.*

Proof. $(\sigma_1, \sigma_2) \mapsto (-\sigma_1, -\sigma_2)$ sends $\mathcal{B}_1 \leftrightarrow \mathcal{B}_2$ and preserves E. This realises the suspension $\Sigma^2 = \mathrm{Id}$ of $D_{\mathrm{sg}}(R)$. □

Corollary 7 (Confinement = Hensel obstruction). *The vacuum sector lifts to $\mathbb{Z}_3$ (the element $(3, 0)$ is a fixed integer). The matter sector does not: the optimal element at precision k changes at every k. The conductor $\mathfrak{c} = 3R$ is the sharp boundary: Hensel lifting converges on $\mathfrak{c}$ and fails on $R \setminus \mathfrak{c}$.*

5 Two-defect interaction

Theorem 8. *Two vacuum excitations $s_a = s_b = (3, 0)$ at distance r on Λ:*

1. *$V(r) = 0$ for $r \geq 2$. The defects are free.*
2. *$V(1) = -72 = -8 \cdot 3^2$ for nearest neighbours, independent of the direction assignment.*

Proof. For $r \geq 2$, no link has both endpoints at defect sites, so $E_{\mathrm{pair}} = 2E_0$ and $V = 0$. For $r = 1$, the two shared directed links change from $\|s\|^2 + \|T_p s\|^2$ to $\|s - T_p s\|^2$ and vice versa. The difference sums to $V = -72$ over all six operators. □

6 The vacuum algebra

In the normalisation, ring multiplication is componentwise: $(\sigma_1, \sigma_2) \cdot (\sigma_1', \sigma_2') = (\sigma_1 \sigma_1', \sigma_2 \sigma_2')$.

Proposition 9. *The boundary class is preserved: products of vacuum elements are vacuum elements. No product of two vacuum excitations produces matter.*

Proof. $\sigma_1 \equiv 0$ and $\sigma_1' \equiv 0$ implies $\sigma_1 \sigma_1' \equiv 0 \pmod 3$. □

Proposition 10 (Sub-threshold non-closure). $(3, 0) \cdot (3, 0) = (9, 0)$ *with $E(9, 0) = 9 \cdot E(3, 0) = 10935$, exceeding the matter threshold. Two light vacuum excitations produce a heavy vacuum excitation.*

Proposition 11 (Cross-branch annihilation). $(3, 0) \cdot (0, 6) = (0, 0)$. *A σ_1-excitation and a σ_2-excitation annihilate to the vacuum.*

7 The boundary grammar

On the boundary $R/3R \cong \mathbb{F}_3$, each Hecke operator acts as multiplication by $a_p(E)$ mod 3:

p	3	7	11	19	43	67
a_p mod 3	0	2	0	0	1	1

Three operators *kill* (×0: T_3, T_{11}, T_{19}), one *swaps* (×2: T_7), and two are the *identity* (×1: T_{43}, T_{67}).

On a plaquette with corners $b_{00}, b_{10}, b_{01}, b_{11} \in \mathbb{F}_3$, the boundary action cost $S_\partial = \sum_{\text{links}} |b_i - (a_p \bmod 3)\cdot b_j|^2$ defines a weighted code on $\mathbb{F}_3^4$. The unique zero-cost pattern is the vacuum $(0,0,0,0)$. The cost function is invariant under the bulk precision k: the boundary grammar is a property of the conductor, not the bulk.

8 The Hecke automaton

The Heegner point $P = (1,0) \in E(\mathbb{Q})$ has formal logarithm $z(P) \in 3\mathbb{Z}_3$ with $z(P)/3 \equiv 1 \pmod 3$ (matter sector). In the formal group $E_1(\mathbb{Q}_3) \cong \mathbb{Z}_3$, the six Hecke operators act by multiplication by $a_p(E)$.

Theorem 12 (Monotone counter machine)**.** *The action of the Hecke eigenvalues* $\{0, 2, -6, -6, 7, -2\}$ *on* $\mathbb{Z}_3$ *by multiplication defines a one-counter automaton with states* $(v, r) \in \mathbb{Z}_{\geq 1} \times \mathbb{F}_3^\times$ *and transitions:*

$$\begin{array}{ll} \text{ANNIHILATE } (a_3 = 0): & (v,r) \to \perp \\ \text{SWAP } (a_7 = 2): & (v,r) \to (v,\, 2r) \\ \text{SHIFT } (a_{11} = a_{19} = -6): & (v,r) \to (v{+}1,\, r) \\ \text{ID } (a_{43} \equiv a_{67} \equiv 1): & (v,r) \to (v,r) \end{array}$$

The counter v *is monotone: it can only increase. The reachable set from the Heegner state* $(1,1)$ *is exactly* $3\mathbb{Z}_3 = \mathfrak{c}$*, the conductor ideal. The matter sector* $\mathbb{Z}_3^\times$ *is unreachable.*

Proof. The unit gates $\{2, 7, -2\}$ generate $(\mathbb{Z}/3^k\mathbb{Z})^\times$ at every k (verified for $k \leq 5$), giving full permutation within each shell. The shift gate $-6 = -2 \cdot 3$ increments v by 1 (the factor -2 is a unit, leaving r unchanged modulo 3). The annihilator sends everything to $\perp$. No gate decrements v, so the matter sector $(v = 0)$ is unreachable. □

Corollary 13. *The Heegner point* $P \in E(\mathbb{Q})$ *is a confined excitation of the Hecke lattice: it generates* $E(\mathbb{Q})$ *but lives in the matter sector of the defect spectrum, with energy diverging as* $\Theta(3^{2k})$*.*

9 Numerical verification

k	3^k	$E_{\text{vac},1}$	E_{matter}	Δ
3	27	324	657	333
4	81	1215	3 087	1 872
5	243	1215	23 445	22 230
6	729	1215	185 877	184 662

The vacuum energy stabilises at $k = 4$. The matter energy grows as $E_m \approx 0.35 \cdot 3^{2k}$. The $\mathbb{Z}/2\mathbb{Z}$ symmetry is exact at every k.

n	$E(3^n,0)$	$E/9^n$
1	1 215	135
2	10 935	135
3	98 415	135

The vacuum tower is exactly geometric with constant $135 = 6 + 3 \cdot 43$.

10 The probabilistic automaton

With uniform gate selection (probability 1/6 each), the automaton becomes a Markov chain.

Theorem 14 (Emergent Boltzmann distribution)**.** *The steady-state depth distribution under uniform gate selection with continuous injection at $v = 1$ is geometric:*

$$\rho(v) \propto q^v, \qquad q = \frac{S}{A+S},$$

where A is the number of annihilator gates and S the number of shift gates. The permutation count P does not enter. For $(A, S, P) = (1, 2, 3)$: $q = 2/3$.

Proof. Conditioned on survival (probability $(S+P)/(A+S+P)$ per step), each step shifts with probability $S/(S+P)$ and permutes otherwise. The depth increment is Bernoulli with parameter $S/(S+P)$. Combined with geometric killing (rate $A/(A+S+P)$), the steady-state depth is geometric with ratio $q = (\text{survival}) \times (\text{shift rate})/(1 - (\text{survival}) \times (\text{1-shift rate})) = S/(A + S)$. □

Corollary 15 (Soliton lifetime = ring precision)**.** *The mean depth at death is $\langle v\rangle = 1 + S/A$. For $(1, 2, 3)$: $\langle v\rangle = 3$, corresponding to the 3-shell ring $R/3^3R = R/27R$.*

The emergent temperature is $T = 1/\ln((A+S)/S) = 1/\ln(3/2)$ for the $(1, 2, 3)$ class. No temperature parameter is imposed: it follows from the gate statistics.

10.1 The light cone

The six Hecke gates classify the lattice directions:

Timelike (permutation, same shell):	$-x$, $\pm y$
Spacelike (shift, deeper shell):	$\pm z$
Null (annihilation):	$+x$

A vacuum soliton $(3, 0)$ propagates freely in the three timelike directions, irreversibly deepens in the two spacelike directions, and dies in the null direction. The signature $(3, 2, 1)$ is determined by the distribution of $v_3(a_p(E))$ over the six Heegner primes. The chirality ($+x$ null, $-x$ timelike) arises from the supersingularity $a_3(E) = 0$.

10.2 Lattice verification

A single defect $(3, 0)$ on a 24^3 lattice at mod 729 spreads with initial depth growth rate

$$\left.\frac{\Delta\langle v\rangle}{\Delta n}\right|_{n=0} = 0.40 = \frac{2}{5} = \frac{S}{S+P},$$

matching the automaton exactly. The depth saturates at $\langle v\rangle \approx 3.5$ due to mod-M wrapping.

10.3 Universality

Among elliptic curves with $a_3(E) = 0$ and prime conductor $N < 1000$, five automaton classes arise:

(A,S,P)	q	$\langle v\rangle$	T	sig.	curves
$(2,0,4)$	0	1	∞	$(4,0,2)$	17, 811
$(1,0,5)$	0	1	∞	$(5,0,1)$	73
$(1,1,4)$	$\frac{1}{2}$	2	$1/\ln 2$	$(4,1,1)$	109
$(1,2,3)$	$\frac{2}{3}$	3	$1/\ln\frac{3}{2}$	$(3,2,1)$	163,179,197,269,739
$(2,1,3)$	$\frac{1}{3}$	$\frac{3}{2}$	$1/\ln 3$	$(3,1,2)$	307

11 Discussion

The lattice model is a sigma model with singular target. The target space $\mathrm{Spec}(R)$ is a nodal curve: two copies of $\mathrm{Spec}(\mathbb{Z}_3)$ meeting at a single point (the conductor). The field $s\colon \Lambda \to \mathrm{Spec}(R)$ assigns to each lattice site a section of the structure sheaf of the node.

The two branches are:

- σ_1: the rational newform $f = f_{V_1}$, attached to the elliptic curve $E\colon y^2 + y = x^3 - 2x + 1$;
- σ_2: the degree-5 eigenform f_{V_5}, defined over $\mathbb{Q}[x]/(x^5 + 5x^4 + 3x^3 - 15x^2 - 16x + 3)$.

The conductor $\mathfrak{c} = 3R$ is the singular point where the branches meet. The vacuum sector consists of sections supported at the node. The matter sectors consist of sections supported on a single branch. Confinement is the obstruction to extending a branch section to a global section of the nodal curve over the full 3-adic ring.

The chain of identifications is:

$$e^{\pi\sqrt{163}} \approx \mathbb{Z} \longrightarrow h(-163) = 1 \longrightarrow E/\mathbb{Q} \longrightarrow f \equiv g \pmod 3 \longrightarrow R \longrightarrow \text{confinement.}$$

11.1 The five numbers

The model is determined by five arithmetic constants:

$$\begin{aligned}
\textstyle\sum a_p(E)^2 &= 129 = 3 \cdot 43 && \text{(vacuum stiffness, Heegner prime)}\\
E_0 &= 1215 = 5 \cdot 3^5 && \text{(vacuum energy)}\\
V(1) &= -72 = -8 \cdot 3^2 && \text{(nearest-neighbour attraction)}\\
E_m/M^2 &\to 0.35\ldots && \text{(confinement constant)}\\
\Sigma^2 &= \mathrm{Id} && (\mathbb{Z}/2\mathbb{Z} \text{ of } D_{\mathrm{sg}}(R))
\end{aligned}$$

11.2 The discrete Poincaré disc

The automaton state (v, r) has a natural polar structure: $v \geq 1$ is the radius (conductor depth), $r \in \mathbb{F}_3^\times$ the angle (branch choice). The Boltzmann factor $(2/3)^v$ defines a discrete metric

$$ds^2 = \left(\tfrac{2}{3}\right)^v dr^2 + dv^2.$$

The angular cost decreases exponentially with depth: swapping branches is cheap in the bulk, expensive near the boundary. Geodesics prefer shallow orbits, but the shift gates push the soliton deeper at rate 2/5 per step. The conductor is a *repulsive force* — the node at $v = 0$ pushes outward.

In the 3-adic limit $k \to \infty$: the soliton drifts to $v = \infty$, angular motion costs nothing, and the soliton becomes a pure radial ray. Confinement is radial escape on the Poincaré disc.

11.3 The soliton

On the lattice, the vacuum excitation $(3,0)$ is *immortal*: it never annihilates (the gate T_3 with $a_3 = 0$ contributes zero to the additive lattice sum, unlike the multiplicative automaton where it absorbs). It oscillates with period 3 in amplitude and breathes with period $9 = 3^2$ in spatial extent, contracting to ~ 5 sites every 9 steps. The breathing period is the conductor squared.

The topological charge is $K_0(D_{\mathrm{sg}}(R)) = \mathbb{Z}/2\mathbb{Z}$: the σ_1-soliton $(3,0)$ carries charge $[B_0] = +1$, the σ_2-soliton $(0,6)$ carries $[B_3] = -1$, and ring multiplication gives annihilation: $(3,0)\cdot(0,6) = (0,0)$.

11.4 The path integral

The amplitude to reach lattice site x after n steps is the sum over all gate-sequences (paths) of length n from the origin to x, weighted by the product of eigenvalues:

$$G(x,n) = \sum_{\text{paths } \gamma:0\to x} \prod_{i=1}^{n} a_{p_i}(E).$$

Each eigenvalue a_p contributes a *direction-dependent mass*:

$$m(p) = \frac{\log|a_p(E)|}{\log 3}.$$

The six masses are:

direction	$+x$	$-x$	$+y$	$-y$	$+z$	$-z$
a_p	0	-2	2	7	-6	-6
$\lvert a_p\rvert$	0	2	2	7	6	6
$v_3(a_p)$	∞	0	0	0	1	1
gate	A	P	P	P	S	S
$m(p)$	∞	0.63	0.63	1.77	1.63	1.63

One table. Three readings. The gate type (v_3) determines the automaton. The mass ($\log|a_p|$) determines the path integral. The eigenvalue itself (a_p) determines the lattice amplitude. All three are the same six integers $\{0, 2, -6, -6, 7, -2\}$, which are the Fourier coefficients of the elliptic curve $E\colon y^2 + y = x^3 - 2x + 1$ at the six Heegner primes.

11.5 Open questions

1. Is $\sum a_p(E)^2 = 3 \cdot 43$ a special value of the symmetric square L-function $L(\mathrm{sym}^2 E, s)$?
2. The Adèlic product over supersingular primes $p \in \{2, 3, 17\}$ gives a multi-tape automaton with constraint $v_3 \le v_2$ (the 3-adic world nested inside the 2-adic world). Is this constraint universal or specific to $E =$ `163a1`?
3. The quantum automaton (Hamiltonian on the state space) has ground state at $v = V_{\max}/2$. The Heegner point multiple $4 \cdot 3^{v-1} \cdot P$ connects the quantum ground state to the formal group. Is this a Gross–Zagier relation?
4. The path integral gives direction-dependent mass $m(p) = \log|a_p|/\log 3$. Is the mass spectrum $\{0.63, 0.63, 1.63, 1.63, 1.77\}$ related to a spectral measure on the Hecke algebra?

5. **The gate lens on neural networks.** Classifying the weights of a trained network (e.g. GPT-2) by magnitude into kill/shift/permute bins reveals depth-dependent structure: shallow layers kill more (A decreasing from 0.24 to 0.12), deep layers shift more (S increasing from 0.29 to 0.43). Is this a universal property of trained deep networks?

6. The five automaton classes (A, S, P) for curves with $a_3 = 0$ and prime conductor $N < 1000$: do they exhaust all possibilities, or are there further classes at larger N?