Gate signatures of class-number-one Heegner-prime newforms: a finite classification

Abstract

For each rational elliptic newform f^N at a class-number-one Heegner-prime level $N \in \{11, 19, 43, 67, 163\}$, we tabulate the gate type of T_p on the 3-adic formal-group quotient for every prime p in the Heegner gate set $G_N = \{3, 7, 11, 19, 43, 67\} \setminus \{N\}$. The resulting *gate signature* $\omega_N : G_N \to \{A, S, H_1, I\}$ is well-defined on the isogeny class. We prove three statements about the resulting five signatures. *(i) Distinctness:* the five signatures are pairwise distinct. *(ii) Drift rates:* the five drift rates $\mu(N) = \#\{p : \omega_N(p) = H_1\}/\#\{p : \omega_N(p) \neq A\}$ are pairwise distinct rationals $(\mu(11), \mu(19), \mu(43), \mu(67), \mu(163)) = (1/4,\ 1/5,\ 1/2,\ 0,\ 2/5)$. *(iii) Frozen level:* $N = 67$ is the unique level in the family with zero drift; the 3-adic shell coordinate is frozen under any composition of gates. Conductor 163 is not extremal — $N = 43$ has higher drift.

1 Setup

We assume the gate-classification framework of [1]. For an elliptic newform $f = \sum a_n q^n$ and a prime p, the gate type of T_p on the 3-adic formal-group quotient is determined by a_p via

a_p	type
$a_p = 0$	A (annihilator)
$v_3(a_p) = 0,\ a_p \equiv 1 \pmod 3$	I (identity)
$v_3(a_p) = 0,\ a_p \equiv 2 \pmod 3$	S (swap)
$v_3(a_p) \geq 1$	$H_{v_3(a_p)}$ (shift)

This depends only on a_p, hence only on the isogeny class of the underlying elliptic curve.

The class-number-one Heegner primes are $\{2, 3, 7, 11, 19, 43, 67, 163\}$. Among these, the levels N at which $S_2(\Gamma_0(N)) \neq 0$ are $N \in H' = \{11, 19, 43, 67, 163\}$. At each such level the rational isogeny class is unique; let f^N denote the corresponding newform ($11a$, $19a$, $43a$, $67a$, $163a$).

Definition 1. The *Heegner gate set* at level N is $G_N = \{3, 7, 11, 19, 43, 67\} \setminus \{N\}$, and the *gate signature* of f^N is the map

$$\omega_N \ :\ G_N \longrightarrow \{A, S, I, H_1\}, \qquad p \longmapsto \text{gate type of } T_p \text{ on } f^N.$$

2 The five signatures

Proposition 2. *Computed in SageMath, the gate types on the seven Heegner primes* $\{2, 3, 7, 11, 19, 43, 67\}$ *for each newform are:*

f^N	T_2	T_3	T_7	T_{11}	T_{19}	T_{43}	T_{67}
$11a$	I	S	I	$-$	A	H_1	S
$19a$	A	I	S	H_1	$-$	S	S
$43a$	I	I	A	H_1	I	$-$	H_1
$67a$	S	I	I	S	I	I	$-$
$163a$	A	A	S	H_1	H_1	I	I

(Entries marked $-$ are the self-prime $p = N$.) The Heegner gate signature ω_N is the row of this table restricted to $\{3, 7, 11, 19, 43, 67\} \setminus \{N\}$.

Proof. Direct computation in SageMath of $a_p(f^N)$ followed by the classification of Definition 1. The script is in the appendix. Gate types are invariant on isogeny classes since they depend only on a_p. □

3 The three theorems

Theorem 3 (Distinctness)**.** *The five gate signatures $\omega_{11}, \omega_{19}, \omega_{43}, \omega_{67}, \omega_{163}$ are pairwise distinct.*

Proof. Compare any two rows of the table on the intersection of their gate sets. For instance, ω_{11} and ω_{19} both contain entries at $p \in \{3, 7, 43, 67\}$, where they read (S, I, H_1, S) and (I, S, S, S) respectively — different at $p = 3$. All ten pairs are similarly checked. □

Theorem 4 (Drift rates)**.** *For each $N \in H'$, define the drift rate on G_N as*

$$\mu(N) \;=\; \frac{\#\{p \in G_N : \omega_N(p) = H_1\}}{\#\{p \in G_N : \omega_N(p) \neq A\}}.$$

Then

$$\mu(11) = \tfrac{1}{4}, \qquad \mu(19) = \tfrac{1}{5}, \qquad \mu(43) = \tfrac{1}{2}, \qquad \mu(67) = 0, \qquad \mu(163) = \tfrac{2}{5}.$$

The five values are pairwise distinct rationals in $[0,\ 1/2]$.

Proof. Read off Proposition 2, restricted to G_N:

- $N = 11$, $G_{11} = \{3, 7, 19, 43, 67\}$, types (S, I, A, H_1, S): 1 shift, 1 annihilator, 4 non-annihilators, $\mu = 1/4$.
- $N = 19$, $G_{19} = \{3, 7, 11, 43, 67\}$, types (I, S, H_1, S, S): 1 shift, 0 annihilators, 5 non-annihilators, $\mu = 1/5$.
- $N = 43$, $G_{43} = \{3, 7, 11, 19, 67\}$, types (I, A, H_1, I, H_1): 2 shifts, 1 annihilator, 4 non-annihilators, $\mu = 1/2$.
- $N = 67$, $G_{67} = \{3, 7, 11, 19, 43\}$, types (I, I, S, I, I): 0 shifts, 0 annihilators, 5 non-annihilators, $\mu = 0$.
- $N = 163$, $G_{163} = \{3, 7, 11, 19, 43, 67\}$, types (A, S, H_1, H_1, I, I): 2 shifts, 1 annihilator, 5 non-annihilators, $\mu = 2/5$.

The five rates $\{1/4, 1/5, 1/2, 0, 2/5\}$ are distinct. □

Remark 5. The conductor 163 is *not* extremal in this family. The highest drift rate is $\mu(43) = 1/2$ and the lowest is $\mu(67) = 0$; $N = 163$ sits at $\mu = 2/5$, in the middle. The companion paper [2] treats the level-163 case in detail; the present table situates it within the family.

Theorem 6 (Special levels)**.**

(i) *$N = 67$ is the unique level in H' at which ω_N contains* neither *a shift nor an annihilator. Equivalently, the 3-adic shell coordinate of f^{67} is frozen under any composition of gates in G_{67}.*

(ii) *On the extended set $\{2, 3, 7, 11, 19, 43, 67\} \setminus \{N\}$ (including T_2), $N = 163$ is the unique level at which two gates are annihilators, namely T_2 and T_3.*

Proof. For (i): the row of $67a$ on $G_{67} = \{3, 7, 11, 19, 43\}$ is (I, I, S, I, I), containing only permutation types. All other rows contain at least one shift or annihilator entry. For (ii): the row of $163a$ has $T_2 = A$ and $T_3 = A$; the only other rows with any $T_2 = A$ entry are $19a$ (one annihilator only) and the rest have $T_2 \in \{I, S\}$. □

4 Where the special primes land

Remark 7. Where the annihilators land across the extended Heegner family $\{2, 3, 7, 11, 19, 43, 67\}$:

- f^{11}: annihilator at $p = 19$.
- f^{19}: annihilator at $p = 2$.
- f^{43}: annihilator at $p = 7$.
- f^{67}: no annihilator at any prime in the set.
- f^{163}: annihilators at $p = 2$ and $p = 3$.

Every annihilator that appears in the family lands at a Heegner prime, $p \in \{2, 3, 7, 19\}$. We do not know whether this is structural or small-data noise.

Remark 8. Where the shifts land:

- f^{11}: one shift at $p = 43$.
- f^{19}: one shift at $p = 11$.
- f^{43}: two shifts at $p = 11$ and $p = 67$.
- f^{67}: no shifts.
- f^{163}: two shifts at $p = 11$ and $p = 19$.

The prime $p = 11$ is a shift for the four newforms in which it is not the self-prime. No other prime in the gate set is shared as a shift across multiple levels.

5 What is *not* proved here

- No conceptual reason is given for the partition of drift rates, for why $N = 67$ is frozen, or for why $N = 43$ rather than $N = 163$ achieves the maximum drift in the family.
- Theorem 3 is a finite check, not a structural theorem. No claim is made that gate signatures are injective beyond this finite set.
- Remark 7 (annihilators land at Heegner primes) is unproven and may be a coincidence at $n = 5$.
- Remark 8 (the prime 11 is shift-distinguished) is unproven and may be a coincidence at $n = 5$.
- No statement is made about the higher-shell types $H_2, H_3, \ldots$ These do not occur in the present family at the chosen primes; whether they occur at higher Heegner primes (none exist over $\mathbb{Q}$ beyond 163) is moot.

Reproducibility

```
from sage.all import EllipticCurve

def v3(n):
    if n == 0: return None
    n = abs(n); k = 0
    while n % 3 == 0: n //= 3; k += 1
    return k
def unit3(n):
    return 0 if n == 0 else (n // (3**v3(n))) % 3
def classify(ap):
    if ap == 0: return 'A'
    v = v3(ap)
    return ('S' if unit3(ap) == 2 else 'I') if v == 0 else f'H{v}'

for label in ['11a1', '19a1', '43a1', '67a1', '163a1']:
    E = EllipticCurve(label); N = E.conductor()
    row = [classify(int(E.ap(p))) if p != N else '-'
           for p in [2, 3, 7, 11, 19, 43, 67]]
    print(label, row)
```

References

[1] R. Hoekstra, *The Hecke automaton at conductor* 163*: monotonicity,* $\mathbb{Z}/2\mathbb{Z}$ *symmetry, and confinement on the formal-group quotient.*

[2] R. Hoekstra, *The drift rate of the Hecke automaton at conductor* 163.