HECKE TRACE GEOMETRY AND THE COTANGENT LAPLACIAN AT HEEGNER PRIMES OF LEVEL 163

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Abstract. Let $G = \big(\mathrm{Tr}(T_d T_{d'} \mid S_2(\Gamma_0(163)))\big)_{d,d' \in S}$ be the Hecke trace Gram matrix restricted to the set $S = \{3, 7, 11, 19, 43, 67, 163\}$ of Heegner primes at level 163. We prove that the Hecke distance matrix $D(d, d') = G(d, d) + G(d', d') - 2G(d, d')$ determines the centered Hecke operator $B = JGJ$ exactly via the classical double-centering identity. We then construct the cotangent Laplacian L_{cot} of the complete graph K_7 equipped with edge lengths $\ell(d, d') = \sqrt{D(d, d')}$, and show computationally that L_{cot} is nearly simultaneously diagonalizable with B: the relative commutator norm is 0.030, eigenvector overlaps range from 0.931 to 0.997, and the relative off-diagonal Frobenius fraction of L_{cot} in a B-eigenbasis is 0.071. A cumulative family comparison across the levels $\{43, 67, 163\}$ shows that this near-diagonalizability is strongest at level 163 among those three cases. A broader scan up to prime level 199, however, shows that level 163 is not globally extremal for this metric once larger cumulative families are included.

1. Introduction

The space $S_2(\Gamma_0(163))$ of weight-2 cusp forms for the congruence subgroup $\Gamma_0(163)$ has dimension $g = 13$, where g is the genus of the modular curve $X_0(163)$. The prime $p = 163$ is distinguished as the largest prime for which $\mathbb{Q}(\sqrt{-p})$ has class number one, a list completed by Heegner [2], Stark [8], and Baker [1].

For each prime d in the set

$$S = \{3,\ 7,\ 11,\ 19,\ 43,\ 67,\ 163\},$$

the Hecke operator T_d acts on $S_2(\Gamma_0(163))$. (At $d = 163$, the operator is $T_{163} = U_{163}$, since 163 divides the level.) The Hecke trace Gram matrix

$$G(d, d') = \mathrm{Tr}(T_d T_{d'} \mid S_2(\Gamma_0(163))) \tag{1}$$

is a 7×7 symmetric positive definite matrix with integer entries, computable via the Eichler–Selberg trace formula or directly in a computer algebra system supporting modular forms (we used SageMath [6]).

The present note has two layers.

First, we record the exact distance-geometric identity

$$B := JGJ = -\tfrac{1}{2} JDJ,$$

where $D(d, d') = G(d, d) + G(d', d') - 2G(d, d')$ and $J = I_7 - \frac{1}{7}\mathbf{1}\mathbf{1}^T$ is the centering matrix. This is the classical double-centering identity of Schoenberg [7], but here it identifies the centered Hecke operator with the canonical centered Gram matrix recovered from Hecke trace distances.

Second, we compare B to the cotangent Laplacian L_{cot} built from the same distance data. The complete graph K_7 endowed with edge lengths $\ell(d, d') = \sqrt{D(d, d')}$ carries a canonical formal cotangent operator. Although this graph is not a triangulated surface in the usual Regge sense, the resulting symmetric Laplacian is numerically almost diagonal in a B-eigenbasis. This suggests that the Hecke trace geometry and the discrete cotangent geometry are much closer than one would expect a priori. The level 163 example is the main case study; a later scan is included only to show that the phenomenon is broader than a single isolated example.

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2. The Hecke trace Gram matrix

All computations were performed in SageMath using `ModularSymbols(163, 2)` and verified by independent evaluation of traces.

Proposition 1. *The matrix* $G = \big(\mathrm{Tr}(T_d T_{d'} \mid S_2(\Gamma_0(163)))\big)_{d,d' \in S}$*, with rows and columns ordered as* $(3, 7, 11, 19, 43, 67, 163)$*, is*

$$G = \begin{pmatrix} 46 & -18 & -12 & 20 & -44 & -70 & 6 \\ -18 & 92 & -10 & -44 & -4 & -60 & 4 \\ -12 & -10 & 136 & -38 & -88 & 120 & 6 \\ 20 & -44 & -38 & 282 & -102 & 6 & -6 \\ -44 & -4 & -88 & -102 & 583 & -164 & -13 \\ -70 & -60 & 120 & 6 & -164 & 640 & -14 \\ 6 & 4 & 6 & -6 & -13 & -14 & 13 \end{pmatrix}.$$

Its eigenvalues, in increasing order, are approximately

$$10.37,\ 19.44,\ 76.78,\ 98.09,\ 274.14,\ 491.62,\ 821.55.$$

In particular, G *is positive definite and* $G(163, 163) = 13 = g$*.*

Proof. Direct computation in SageMath. The identity $G(163, 163) = \mathrm{Tr}(T_{163}^2 \mid S_2(\Gamma_0(163))) = \mathrm{Tr}(U_{163}^2) = \dim S_2(\Gamma_0(163)) = 13$ follows from the fact that $U_{163} = -w_{163}$ (the Atkin–Lehner involution) at prime level, so $U_{163}^2 = w_{163}^2 = \mathrm{Id}$ on the 13-dimensional space. □

3. The double-centering identity

Definition 2. Given the Gram matrix G, define the *Hecke distance matrix* by

$$D(d, d') = G(d, d) + G(d', d') - 2\,G(d, d'),$$

and the *centering matrix*

$$J = I_7 - \tfrac{1}{7}\,\mathbf{1}\mathbf{1}^T,$$

where $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^7$. The *centered Hecke operator* is $B = JGJ$.

Theorem 3 (Double-centering identity)**.** *With* G*,* D*,* J*, and* B *as above,*

$$B = JGJ = -\tfrac{1}{2}\,JDJ. \tag{2}$$

Both equalities hold exactly over $\mathbb{Q}$*.*

Proof. Write $D = \mathbf{g}\,\mathbf{1}^T + \mathbf{1}\,\mathbf{g}^T - 2G$, where $\mathbf{g} = \mathrm{diag}(G) = (G(d, d))_{d \in S}$. Then

$$-\tfrac{1}{2}\,JDJ = -\tfrac{1}{2}\,J\big(\mathbf{g}\,\mathbf{1}^T + \mathbf{1}\,\mathbf{g}^T - 2G\big)J = JGJ,$$

since $J\mathbf{1} = \mathbf{0}$ and $\mathbf{1}^T J = \mathbf{0}^T$. This is Schoenberg's classical identity [7]. Since all entries of G and D are rational integers, the identity holds exactly over $\mathbb{Q}$. □

Remark 4. The matrix B annihilates $\mathbf{1}$, hence $\mathrm{rank}(B) \le 6$. In the present case B has exactly six nonzero eigenvalues, namely

$$10.632739,\ 60.562594,\ 96.738276,\ 251.545366,\ 455.849771,\ 810.671253,$$

and one zero eigenvalue in the direction of $\mathbf{1}$.

4. The cotangent operator

We equip the complete graph K_7 on vertex set S with edge lengths

$$\ell(d, d') = \sqrt{D(d, d')}.$$

Definition 5. For each triangle (d, d', d'') in K_7, define the angle at vertex d by the Euclidean law of cosines:

$$\cos \alpha_d(d', d'') = \frac{\ell(d,d')^2 + \ell(d,d'')^2 - \ell(d',d'')^2}{2\,\ell(d,d')\,\ell(d,d'')}.$$

The *cotangent weight* of edge (d, d') is

$$w(d,d') = \frac{1}{2} \sum_{\substack{d'' \in S \\ d'' \neq d, d'}} \cot \alpha_{d''}(d, d'),$$

where the sum runs over the five triangles containing the edge (d, d').

The *cotangent Laplacian* is the 7×7 matrix

$$L_{\cot}(d,d') = \begin{cases} -w(d,d') & d \neq d', \\ \displaystyle\sum_{d'' \neq d} w(d,d'') & d = d'. \end{cases}$$

Remark 6. The complete 2-skeleton of K_7 is not a 2-manifold triangulation: each edge lies in five triangles. Thus $L_{\cot}$ should be read as the formal cotangent operator attached to the Hecke length data, rather than as the Laplace–Beltrami operator of a genuine piecewise-flat surface.

Remark 7. The matrix $L_{\cot}$ is symmetric and satisfies $L_{\cot}\,\mathbf{1} = \mathbf{0}$. In the present computation its spectrum is nonnegative:

$$0,\ 5.250892,\ 7.378571,\ 10.591287,\ 15.303978,\ 17.814893,\ 26.837106.$$

One cotangent edge weight is slightly negative, so we do not claim positive semidefiniteness by abstract construction alone.

5. Near-simultaneous diagonalizability

The exact theorem concerns B, not G. Indeed, an identity $L_{\cot} = G$ is impossible, because

$$L_{\cot}\,\mathbf{1} = \mathbf{0}, \qquad G\,\mathbf{1} = (-72,\ -40,\ 114,\ 118,\ 168,\ 458,\ -4)^T.$$

Thus the correct comparison is between $L_{\cot}$ and the centered Hecke operator B on $\mathbf{1}^{\perp}$.

Proposition 8 (Computational)**.** *Let B and $L_{\cot}$ be as above, restricted to the six-dimensional subspace $\mathbf{1}^{\perp}$. Then:*

(a) The sorted nonzero eigenvalue sequences of B and $L_{\cot}$ have Pearson correlation coefficient

$$\rho = 0.982148.$$

(b) Under the overlap-maximizing matching of eigenvectors, the pairing is order-reversing: the largest eigenvalue of B is matched with the smallest nonzero eigenvalue of $L_{\cot}$, and vice versa.

(c) Each eigenvector v_i of B on $\mathbf{1}^{\perp}$ has a matched eigenvector $u_{\sigma(i)}$ of $L_{\cot}$ such that

$$|\langle v_i,\ u_{\sigma(i)}\rangle| \geq 0.930724,$$

with the maximum overlap attaining 0.996905 *and the mean overlap equal to* 0.973626*.*

(d) The relative commutator norm satisfies

$$\frac{\|[B,\ L_{\cot}]\|_F}{\|B\|_F\ \|L_{\cot}\|_F} = 0.030399,$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

(e) In the eigenbasis of B, the matrix $L_{\cot}$ has off-diagonal Frobenius norm equal to 7.135% *of its total Frobenius norm.*

All values are computed from the exact integer matrix G of Proposition 1, with cotangent weights evaluated in double-precision floating point.

Remark 9. The numerical evidence therefore supports not the false identity $L_{\text{cot}} = G$, but the weaker and more plausible bridge principle that L_{cot} is nearly diagonal in the centered Hecke eigenbasis.

6. A local cumulative comparison

To compare levels with increasing arithmetic complexity, define the cumulative class-number-one sets

$$S^{(43)} = \{3, 7, 11, 19, 43\},$$
$$S^{(67)} = \{3, 7, 11, 19, 43, 67\},$$
$$S^{(163)} = \{3, 7, 11, 19, 43, 67, 163\}.$$

For each level $p \in \{43, 67, 163\}$ we form the Hecke trace matrix $G^{(p)}$ on the corresponding set $S^{(p)}$, then define $D^{(p)}$, $B^{(p)}$, and $L_{\text{cot}}^{(p)}$ exactly as above.

Proposition 10 (Computational)**.** *The off-diagonal fraction $\delta(p)$—defined as the ratio of the off-diagonal Frobenius norm of $L_{\text{cot}}^{(p)}$ in the eigenbasis of $B^{(p)}$ to the total Frobenius norm of that restricted matrix—takes the values*

Level p	$\|S^{(p)}\|$	$\text{rank}(B^{(p)})$	$\delta(p)$
43	5	3	0.100321
67	6	5	0.138548
163	7	6	0.071352

Among these three cumulative levels, the near-diagonalizability is strongest at level 163.

Remark 11. This comparison is cumulative rather than intrinsic: the sets $S^{(43)}$, $S^{(67)}$, and $S^{(163)}$ are nested prefixes of the class-number-one primes, not the split-Heegner sets at the corresponding levels. We make this explicit because the numerical trend depends on that choice of family.

Remark 12. A broader scan over prime levels $N \leq 199$ with the same cumulative family choice shows that level 163 is *not* globally minimal for the off-diagonal fraction. Among the full seven-operator cumulative cases in that scan, levels 173, 197, and 199 all give smaller values than the level-163 value 0.071352. Thus Proposition 10 should be read as a local comparison among the three displayed cumulative levels, not as a global extremality claim for 163.

7. Discussion

The double-centering identity (Theorem 3) is classical, but its application to the Hecke Gram matrix at Heegner primes yields an exact arithmetic consequence: the pairwise Hecke distances determine the centered Hecke operator without loss.

The cotangent operator attached to the same distance data is not equal to the Hecke operator. Nevertheless, Proposition 8 shows that the two operators almost share an eigenbasis. This is the mathematically robust bridge: not equality of matrices, but near-simultaneous diagonalizability on the centered subspace.

The exact content of this note is therefore asymmetric: the double-centering identity is a theorem, while the Hecke–cotangent bridge is a numerical phenomenon anchored at level 163. The local comparison of Proposition 10 shows that the bridge improves from 43 and 67 to 163 inside one natural cumulative family, but the broader scan shows that this does not extend to a global minimality statement. Any claim that 163 is geometrically extremal must therefore depend on a more refined arithmetic selection principle than the cumulative family alone.

One further numerical feature merits mention. If the nonzero eigenspaces are paired by maximum overlap, then the cotangent eigenvalues are fit substantially better by a decreasing rational function of the centered Hecke eigenvalues than by an affine rescaling. This suggests that the correct next conjecture is not $L_{\text{cot}} \approx aB + bI$, but rather that $L_{\text{cot}}|_{\mathbf{1}^\perp}$ is approximately of the form $f(B|_{\mathbf{1}^\perp})$ for a monotone decreasing scalar function f.

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