

The trace inequality at conductor 163 does not extend uniformly across the class-number-one Heegner-prime family

Abstract

Let $H = \{2, 3, 7, 11, 19, 43, 67, 163\}$ be the set of class-number-one Heegner primes, and let $H' = \{11, 19, 43, 67, 163\}$ be those for which $S_2(\Gamma_0(N)) \neq 0$. For each $N \in H'$ form the trace Gram matrix $G_N(d, d') = \text{Tr}(T_d T_{d'} | S_2(\Gamma_0(N)))$ on the nested gate set $S_N = \{p \in H \setminus \{2\} : p \leq N\}$, with $T_N = U_N = -w_N$ at the prime level. We test whether the level-163 trace inequality

$$|G_N(d, d')| \leq 2 \min(G_N(d, d), G_N(d', d'))$$

holds across the family. The answer is: it holds at $N \in \{11, 43, 163\}$ and *fails* at $N \in \{19, 67\}$. The two failures are mild (worst ratio $14/10 = 1.4$ at $N = 67$), and at the two one-dimensional levels $N \in \{11, 19\}$ the inequality reduces to the elementary statement $\max_p |a_p| \leq 2 \min_p |a_p|$ on the gate set, which happens to hold for $11a1$ and fail for $19a1$. We conclude that the inequality at conductor 163 is not a Heegner-family phenomenon.

1 Setup

Let $H = \{2, 3, 7, 11, 19, 43, 67, 163\}$ be the eight rational primes p for which $\mathbb{Q}(\sqrt{-p})$ has class number one [2]. Set

$$H' = \{N \in H : \dim S_2(\Gamma_0(N)) > 0\} = \{11, 19, 43, 67, 163\}.$$

At each level $N \in H'$ we form the nested gate set

$$S_N = \{p \in H \setminus \{2\} : p \leq N\},$$

i.e. $S_{11} = \{3, 7, 11\}$, $S_{19} = \{3, 7, 11, 19\}$, $S_{43} = \{3, 7, 11, 19, 43\}$, $S_{67} = \{3, 7, 11, 19, 43, 67\}$, and $S_{163} = \{3, 7, 11, 19, 43, 67, 163\}$. The gate T_N is interpreted as the Atkin–Lehner-twisted operator $U_N = -w_N$ at the prime level.

The Hecke trace Gram matrix is

$$G_N(d, d') = \text{Tr}(T_d T_{d'} | S_2(\Gamma_0(N))).$$

At $N = 163$ the inequality

$$|G_N(d, d')| \leq 2 \min(G_N(d, d), G_N(d', d')) \tag{1}$$

holds for all 21 pairs in S_{163} [1]. The present note tests (1) at the four other Heegner-prime levels.

2 Computation

Proposition 1. *The trace Gram matrices on S_N at the five levels of H' , computed in SageMath via modular symbols, are:*

$N = 11$, $S_{11} = \{3, 7, 11\}$:

$$G_{11} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

$N = 19$, $S_{19} = \{3, 7, 11, 19\}$:

$$G_{19} = \begin{pmatrix} 4 & 2 & -6 & -2 \\ 2 & 1 & -3 & -1 \\ -6 & -3 & 9 & 3 \\ -2 & -1 & 3 & 1 \end{pmatrix}$$

$N = 43$, $S_{43} = \{3, 7, 11, 19, 43\}$:

$$G_{43} = \begin{pmatrix} 8 & -4 & -14 & 12 & 2 \\ -4 & 12 & 12 & 0 & -4 \\ -14 & 12 & 27 & -18 & -5 \\ 12 & 0 & -18 & 28 & -2 \\ 2 & -4 & -5 & -2 & 3 \end{pmatrix}$$

$N = 67$, $S_{67} = \{3, 7, 11, 19, 43, 67\}$:

$$G_{67} = \begin{pmatrix} 14 & -4 & 14 & -26 & -10 & 2 \\ -4 & 30 & -6 & 0 & -12 & 0 \\ 14 & -6 & 28 & -54 & 14 & -2 \\ -26 & 0 & -54 & 135 & 2 & -5 \\ -10 & -12 & 14 & 2 & 134 & -14 \\ 2 & 0 & -2 & -5 & -14 & 5 \end{pmatrix}$$

$N = 163$, $S_{163} = \{3, 7, 11, 19, 43, 67, 163\}$: as in [1, Prop. 3.1].

Proof. Direct computation in SageMath using `ModularSymbols(N,2,sign=1).cuspidal_subspace()` and `hecke_matrix`. Reproducible by the script in the appendix. \square

3 Status of the inequality at each level

Theorem 2. *The trace inequality (1) holds at $N \in \{11, 43, 163\}$ and fails at $N \in \{19, 67\}$. Specifically:*

- (i) $N = 11$: holds for all $\binom{3}{2} = 3$ pairs.
- (ii) $N = 19$: fails for the two pairs $(7, 11)$ and $(11, 19)$, where $|G_{19}| = 3$ exceeds the bound 2.
- (iii) $N = 43$: holds for all $\binom{5}{2} = 10$ pairs.
- (iv) $N = 67$: fails for the single pair $(43, 67)$, where $|G_{67}(43, 67)| = 14$ exceeds the bound $2 \min(134, 5) = 10$.
- (v) $N = 163$: holds for all $\binom{7}{2} = 21$ pairs [1].

Proof. Read off the Gram matrices of Proposition 1 and check $|G_N(d, d')|$ against $2 \min(G_N(d, d), G_N(d', d'))$ for each pair. The non-trivial line is at $N = 67$:

$$|G_{67}(43, 67)| = 14, \quad 2 \min(G_{67}(43, 43), G_{67}(67, 67)) = 2 \min(134, 5) = 10.$$

□

4 The dimension-one reduction

At $N \in \{11, 19\}$ the cuspidal space is one-dimensional, so G_N has rank one and decomposes as $G_N(d, d') = a_d(f^N) a_{d'}(f^N)$ where f^N is the unique normalised newform. In this case (1) is equivalent to a statement about the spread of the a_p values.

Proposition 3. *For N with $\dim S_2(\Gamma_0(N)) = 1$, the inequality (1) on the gate set S_N is equivalent to*

$$\max_{p \in S_N} |a_p(f^N)| \leq 2 \min_{p \in S_N} |a_p(f^N)|.$$

Proof. $|G_N(d, d')| = |a_d a_{d'}| = |a_d| \cdot |a_{d'}|$, and $2 \min(G_N(d, d), G_N(d', d')) = 2 \min(a_d^2, a_{d'}^2) = 2 \min(|a_d|, |a_{d'}|)^2$. The inequality $|a_d| |a_{d'}| \leq 2 \min(|a_d|, |a_{d'}|)^2$ rearranges to $\max(|a_d|, |a_{d'}|) \leq 2 \min(|a_d|, |a_{d'}|)$. Quantifying over all pairs gives the global ratio bound. □

Corollary 4. *For $N = 11$, $f^N = 11a_1$ has $(a_3, a_7, a_{11}) = (-1, -2, 1)$, so the absolute values are $(1, 2, 1)$ with ratio 2 — the inequality holds (at the boundary). For $N = 19$, $f^N = 19a_1$ has $(a_3, a_7, a_{11}, a_{19}) = (-2, -1, 3, 1)$, so the absolute values are $(2, 1, 3, 1)$ with ratio $3 > 2$ — the inequality fails on the $(7, 11)$ and $(11, 19)$ pairs.*

Proof. Direct from Proposition 3 and the a_p data computed in SageMath. □

Remark 5. The dimension-one cases show that whether (1) holds at a small level is essentially a question about whether the eigenvalues of the unique newform happen to lie within a factor of 2 of one another on the chosen gate set. This is a property of f^N , not of any Heegner structure. At $N = 11$ it holds (just barely); at $N = 19$ it fails (just barely).

5 What the data shows

Theorem 6. *The level-163 trace inequality [1, Thm. 1.1] is not a uniform property of the class-number-one Heegner-prime family H' . It holds at three of five levels and fails at the other two.*

Proof. Theorem 2. □

Remark 7 (Magnitude of failure). The two failures are mild:

- At $N = 19$, the violating ratio is $|G_{19}|/\text{bound} = 3/2$.
- At $N = 67$, the violating ratio is $|G_{67}(43, 67)|/\text{bound} = 14/10 = 7/5$.

In particular, the constant 2 in (1) cannot be strengthened uniformly, but the family-wide constant could plausibly be replaced by something around $7/5 \cdot 2 = 14/5$ if a uniform statement is desired. We do not pursue this.

Remark 8 (Pattern). The levels where the inequality holds are $N \in \{11, 43, 163\}$ and where it fails are $N \in \{19, 67\}$. We have no conceptual explanation for this partition. The dimensions of the cuspidal spaces are $(1, 1, 3, 5, 13)$ at $(11, 19, 43, 67, 163)$, so the holding set has dimensions $(1, 3, 13)$ and the failing set has dimensions $(1, 5)$. The successive ratios in the holding set $(1 \rightarrow 3 \rightarrow 13)$ are $(3, 13/3)$ and the failing-set ratio $(1 \rightarrow 5)$ is 5, but we record this without interpretation.

6 What is *not* proved here

- No conceptual reason is given for the partition $\{11, 43, 163\} \sqcup \{19, 67\}$.
- No uniform constant $C \geq 2$ is computed for which $|G_N(d, d')| \leq C \min(G_N(d, d), G_N(d', d'))$ holds across all of H' . The data of Theorem 2 show $C \geq 7/5 \cdot 2$ is necessary; we do not show it is sufficient at all levels (in particular, larger gate sets at higher levels could produce worse ratios).
- No statement is made about non-prime N , about non-Heegner levels, or about the same construction at higher weight.
- The negative result of Theorem 6 should be read as a statement about the *specific* inequality $|G| \leq 2 \min(G_{dd}, G_{d'd'})$. Variants with different constants or different gate sets are not addressed.

Reproducibility

The complete calculation of all Gram matrices and the inequality check is the following SageMath script:

```
heegner_primes = [3, 7, 11, 19, 43, 67, 163]
levels = [11, 19, 43, 67, 163]

def gate_set(N):
    return [p for p in heegner_primes if p <= N]

def trace_gram(N, S):
    M = ModularSymbols(N, 2, sign=1).cuspidal_subspace()
    n = len(S)
    G = matrix(ZZ, n, n)
    for i, d in enumerate(S):
        Td = M.hecke_matrix(d)
        for j, dp in enumerate(S):
            G[i,j] = (Td * M.hecke_matrix(dp)).trace()
    return G

for N in levels:
    S = gate_set(N)
    G = trace_gram(N, S)
    n = len(S)
    violations = [(S[i], S[j], abs(G[i,j]), 2*min(G[i,i], G[j,j]))
                  for i in range(n) for j in range(i+1, n)
                  if abs(G[i,j]) > 2*min(G[i,i], G[j,j])]
    print(N, "violations:", violations)
```

References

- [1] R. Hoekstra, *A trace inequality for Hecke operators at class-number-one Heegner primes of level 163*.

- [2] H. M. Stark, *A complete determination of the complex quadratic fields of class-number one*, Michigan Math. J. 14 (1967), 1–27.