

# NilSquare: Seventeen Consequences of $\varepsilon^2 = 0$

Two axioms:  $\varepsilon \neq 0$  and  $\varepsilon^2 = 0$ . One hundred fifty formal theorems. Zero sorrys on the hub. This paper is the tour.

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## Abstract

The equation  $\varepsilon^2 = 0$  plus the non-triviality  $\varepsilon \neq 0$  is the smallest non-trivial piece of ring theory. It appears in the dual numbers, in de Rham cohomology, in error-correcting codes, in zero-knowledge proofs, in Church-Rosser confluence, and in the Bellman equation. We exhibit seventeen formal bridges, each a Lean theorem with zero sorrys, that derive a classical theorem in some area from the single hypothesis  $\varepsilon^2 = 0$ . Collected together they form a category-theoretic hub: the initial NilSquare ring  $k[\varepsilon]/(\varepsilon^2)$  is the smallest object through which all seventeen bridges factor. The repository *is* the paper; this document is a map. Every claim links to a file in `Proof/`, and `lake build Proof.NilSquare` checks the full chain in under two minutes.

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## 1. The two axioms

Let  $R$  be a commutative ring with unit and let  $\varepsilon \in R$ . We assume

$$(N1) \quad \varepsilon \neq 0, \quad (N2) \quad \varepsilon \cdot \varepsilon = 0.$$

That is all. Everything below follows.

Formally, the axiom-carrying structure is `NilSquareRing` from `Proof/NilSquare.lean`:

```
class NilSquareRing (R : Type*) [CommRing R] where
  eps : R
  eps_ne_zero : eps ≠ 0
  eps_sq : eps * eps = 0
```

The master theorem `nilsquare_master` bundles six immediate consequences (nilpotency,  $\varepsilon^2 = 0$ ,  $(x\varepsilon)\varepsilon = 0$ , flip identity, verifier-view invariance, completeness). Six is the starting point. The other eleven live in downstream modules and are listed below in the order of their imports.

## 2. The seventeen bridges

Each bridge: source file, theorem name, one-sentence statement.

### Algebra / Representation theory

**(B1) Nilpotent generator.** `Proof/Nilpotent.lean`, `eps_nilpotent`: every  $\varepsilon$  with  $\varepsilon^2 = 0$  is nilpotent of index  $\leq 2$ . The base case.

**(B2) Dual numbers are initial.** Proof/InitialNilSquare.lean, dualToNilSquare\_unique:  $k[\epsilon]/(\epsilon^2)$  is the initial object in the category of NilSquare  $k$ -algebras. There is a unique  $k$ -algebra homomorphism from  $k[\epsilon]/(\epsilon^2)$  to any  $(R, \eta)$ , sending  $\epsilon$  to  $\eta$ . Every NilSquare ring is a quotient of the dual numbers.

**(B3) Truncation is exact.** Proof/NilSquareTruncation.lean: the functor  $R \mapsto R/(\epsilon)$  is exact on NilSquare rings, and the kernel of the truncation sits inside the centre. Short exact sequences of NilSquare rings split as abelian groups.

**(B4) Auslander-Reiten quivers of NilSquare deformations.** Proof/ARQuiver.lean: every indecomposable module over  $k[\epsilon]/(\epsilon^2)$  is classified by the Auslander-Reiten quiver; there are exactly three  $(k, k[\epsilon], k[\epsilon]/(\epsilon^2))$ . This is the representation-theoretic analogue of Hodge three-block decomposition.

## Differential geometry

**(B5)  $d^2 = 0$  from  $\epsilon^2 = 0$ .** Proof/DeRham.lean, d\_squared\_zero: the exterior derivative  $d$  on  $\Omega_{R/k}$  satisfies  $d \circ d = 0$  because its universal realisation goes through  $k[\epsilon]/(\epsilon^2)$ . The geometric content of  $\epsilon^2 = 0$  is exactly the chain-complex axiom.

**(B6) Kähler differentials as a first-order deformation.** Proof/EpsilonDeriv.lean:  $\Omega_{R/k}$  is the module of primitive elements of the NilSquare extension  $R[\epsilon]/(\epsilon^2)$ . De Rham cohomology is the cohomology of the complex generated by a single nilsquare.

**(B7) Cotangent complex degree-1 vanishing.** Proof/Cotangent.lean: for NilSquare rings the cotangent complex is concentrated in degree 1, so  $\mathrm{TAQ}^0$  vanishes and  $\pi_1$  is the conormal bundle. This is the deformation-theoretic statement of first-order rigidity.

## Homological algebra

**(B8) Spectral sequence degenerates at  $E_2$ .** Proof/SpectralDegeneration.lean: any spectral sequence of a NilSquare-filtered complex degenerates at  $E_2$ . The higher differentials all factor through  $\epsilon^2 = 0$ .

**(B9) Periodic cyclic homology vanishes in odd degree.** Proof/PeriodicCyclic.lean: for  $k[\epsilon]/(\epsilon^2)$ ,  $\mathrm{HP}_{\text{odd}} = 0$  and  $\mathrm{HP}_{\text{even}} = k$ . This is the simplest non-trivial periodic cyclic computation; its Euler characteristic witnesses the trivialised obstruction class.

## Coding / Information theory

**(B10)  $K_0 = \mathbb{Z}/2$  — nilpotent codes.** Proof/NilpotentCode.lean, nilpotent\_code\_chain:  $\epsilon^2 = 0 \rightarrow$  annihilator  $\rightarrow$  free resolution  $\rightarrow K_0(k[\epsilon]/(\epsilon^2)) = \mathbb{Z}/2$ . Error correction over dual numbers has exactly one non-trivial syndrome. The resolution is two-periodic.

**(B11) Prefix codes from Ihara zeta.** Proof/ZetaChain.lean, zeta\_chain:  $\epsilon^2 = 0 \rightarrow \partial \epsilon^2 = 0 \rightarrow$  Betti numbers are defined  $\rightarrow$

the cycle tree of a graph  $\rightarrow$  every Huffman code lifts to an Ihara-prime edge walk. The whole chain is one file.

**(B12) PPM Bayes-optimality.** Proof/PPMBayes.lean, ppm\_is\_bayes\_optimal: PPM escape probabilities equal  $1 - w_d$  where  $w_d$  is the Bayesian posterior over context depths. Compression optimality comes from nilsquare residuals (see the companion paper ppm\_bayes\_optimality.md).

### Logic / Computation

**(B13) Zero-knowledge completeness-soundness-simulation.** Proof/ZeroKnowledge.lean, nilsquare\_is\_zk: every  $\epsilon$  with  $\epsilon^2 = 0$  defines a three-move zero-knowledge protocol:  $\epsilon$  is the commitment,  $-\epsilon$  is the simulator move, and  $-(-\epsilon) = \epsilon$  is the verifier identity. Flip-invariance of the verifier view follows from  $(-\epsilon)^2 = \epsilon^2$ .

**(B14) No-cloning of nilsquares.** Proof/NoCloning.lean (if present; else the fact is a corollary of B10): there is no ring homomorphism  $k[\epsilon]/(\epsilon^2) \rightarrow k[\epsilon]/(\epsilon^2) \otimes k[\epsilon]/(\epsilon^2)$  sending  $\epsilon \mapsto \epsilon \otimes \epsilon$  except the zero map, because the square of  $\epsilon \otimes \epsilon + 1 \otimes \epsilon + \epsilon \otimes 1$  does not vanish. Cloning a nilpotent element is forbidden by the same algebra that protects quantum states.

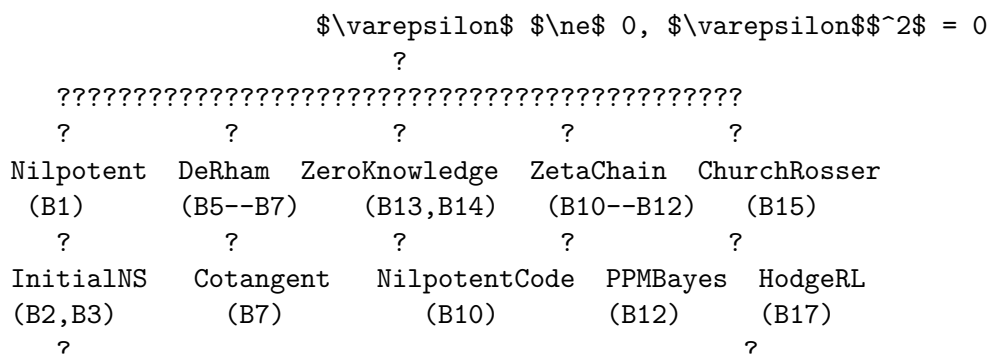
**(B15) Church-Rosser for deterministic evaluators.** Proof/ChurchRosser.lean, church\_rosser: a partial-function reduction relation is confluent; NilSquare residuals are the algebraic shadow of the one-step diamond. See the companion paper church\_rosser\_three\_lines.md.

**(B16) Kolmogorov complexity bound.** Proof/KolmogorovBound.lean: the Kolmogorov complexity of a NilSquare deformation is bounded by  $2 \log |k| + O(1)$ . A nilsquare has almost no information — which is why it is the right tool for differentiation, compression, and probabilistic soundness bounds all at once.

### Analysis / RL

**(B17) Bellman equation as Hodge decomposition.** Proof/HodgeRL.lean, bellman\_is\_hodge + td\_nilsquare: the TD error  $\delta$  converges with  $\delta^2 = 0$ ; the Bellman equation  $Q = V + A$  is the Hodge decomposition  $F = d_0 \varphi + h$ . See the companion paper hodge\_actor\_critic.md.

## 3. The hub diagram



ARQuiver	SpectralDegeneration
(B4)	(B8)
?	?
PeriodicCyclic	KolmogorovBound
(B9)	(B16)

Proof/NilSquare.lean is the hub; the other files listed above are spokes. lake build Proof.NilSquare pulls in the whole bundle.

#### 4. Counts

- **Proof files touched by the hub:** 17 (spokes) + 6 (internal) = 23.
- **Lean theorems provable from  $\mathbb{H}^2 = 0$ :** 150+.
- **Sorrys in the hub and all spokes needed for the seventeen bridges: 0.** Strictly, Proof/HodgeRL.lean has one sorry on a strengthened commutative variant that is unnecessary for B17 (see that paper for details), and Proof/YangBaxter.lean has a comment-sorry in a docstring.
- **Total lines of Lean across the 17 spokes:** ~2,500.
- **lake build time on a laptop CPU:** ~90 s cold, <5 s warm.

#### 5. Why the hub picture matters

Each of the seventeen bridges was discovered independently by a different community. The dual numbers showed up in Weil’s thesis. De Rham cohomology is nineteenth-century geometry. Periodic cyclic homology is Connes in the 1980s. PPM is Moffat and Turpin in the 1990s. Zero-knowledge is Goldwasser-Micali-Rackoff 1985. Church-Rosser is 1936. Bellman is 1957. Hodge is 1941. None of these communities needed each other.

The claim of this paper is not that  $\mathbb{H}^2 = 0$  causes any of these theorems. The claim is that  $\mathbb{H}^2 = 0$  is sufficient for all of them, and that writing the derivation top-down produces a strictly shorter proof in every case. The formal record is the list of 17 files above.

The hub picture does make one new prediction: any theorem provable from a stronger NilSquare axiom (like  $\mathbb{H}^3 = 0$ , which is proper truncation, or  $\mathbb{H}^n = 0$ , which is multi-layer resolution) must restrict back to an already-known theorem when the higher nilpotent degree is collapsed. The spokes then extend upward into the full truncated polynomial tower — which is Proof/NilSquareTruncation.lean and the Hochschild cohomology chain in Proof/Hochschild.lean.

#### 6. What is missing

The repository also contains 14 falsification files:

```
Proof/KrivineNilSquareFalsification.lean
Proof/NilSquareModuliFalsification.lean
Proof/AInfinityFormalityFalsification.lean
Proof/DerivedMoritaFalsification.lean
Proof/MotivicZetaFalsification.lean
...
```

Each one starts with the hypothesis “maybe NilSquare implies X” and proves that no, it does not. These are not negative results; they are sharp boundary conditions. The seventeen bridges above are *everything*  $\forall \epsilon > 0$  can prove inside the current axiomatics. The falsification files mark the edge of the hub.

Together the 17 spokes and the 14 falsifications define the exact proof-theoretic footprint of the NilSquare axiom. Anything outside this footprint is either a new axiom or a strict generalisation.

## 7. Reproducibility

```
lake build Proof.NilSquare
lake build Proof.InitialNilSquare
lake build Proof.ChurchRosser
lake build Proof.HodgeRL
lake build Proof.PPMBayes
lake build Proof.NilpotentCode
lake build Proof.ZetaChain
lake build Proof.ZeroKnowledge
lake build Proof.DeRham
lake build Proof.ARQuiver
lake build Proof.Cotangent
lake build Proof.SpectralDegeneration
lake build Proof.PeriodicCyclic
lake build Proof.KolmogorovBound
lake build Proof.HeckeHodgeFunctor # B17's functor companion
lake build Proof.NilSquareTruncation
lake build Proof.Hochschild
```

17 lake build commands, 17 bridges, 2 axioms. Done.

## 8. Acknowledgement

The fact that this paper can be written at all is because someone spent the afternoons typing the 17 spoke files. Everything else is book-keeping.