THE SINGULAR 3-ADIC HECKE NODE AT LEVEL 163: PRESENTATION, IDEAL THEORY, AND MODULE CLASSIFICATION

RICHARD A. HOEKSTRA

Abstract. We give a complete local description of the singular factor of the Hecke algebra $\mathbf{T}_{163}$ at the prime 3. The full Hecke algebra of weight-2 cusp forms on $\Gamma_0(163)$ has a unique non-semisimple 3-adic local factor, which we identify explicitly as

$$O_{\text{sing}} \;\cong\; \mathbf{Z}_3 \times \mathbf{Z}_3[\eta]/(\eta^2 - 3\eta).$$

The pair factor $R_{\text{pair}} = \mathbf{Z}_3[\eta]/(\eta^2 - 3\eta)$ is the completed local ring of a split nodal curve over $\mathbf{Z}_3$: two $\mathbf{Z}_3$-branches glued at a single residue point. We classify all ideals of the finite quotients (obtaining a $\mathbf{P}^1(\mathbf{F}_3)$-bouquet at each depth), compute the local zeta function, determine the residue representation theory (two simples, one nontrivial self-extension), and classify all torsion-free modules (three indecomposables: the two branches and the node itself). The Heegner discriminant operators T_d for $d \in \{3, 7, 11, 19, 43, 67, 163\}$ are identified explicitly in this presentation, with the syndrome algebra $\mathbf{F}_3 \times \mathbf{F}_3[\varepsilon]/(\varepsilon^2)$ arising as the exact mod-3 shadow.

Contents

1. Introduction

Let $N = 163$. The space $S_2(\Gamma_0(163))$ of weight-2 cusp forms has dimension 13. The Hecke algebra

$$\mathbf{T}_{163} = \mathbf{Z}[T_1, T_2, T_3, \ldots] \subset \operatorname{End}(S_2(\Gamma_0(163)))$$

is a free $\mathbf{Z}$-module of rank 13. Its discriminant is

$$\operatorname{disc}(\mathbf{T}_{163}) = 2^{15} \cdot 3^2 \cdot 65657 \cdot 82536739.$$

The 3-adic valuation $v_3(\operatorname{disc}) = 2$ signals that $\mathbf{T}_{163} \otimes \mathbf{Z}_3$ is not a maximal order. The purpose of this paper is to give a complete description of the singular locus.

Over $\mathbf{Q}_3$, the algebra $\mathbf{T}_{163} \otimes \mathbf{Q}_3$ splits as a product of fields corresponding to the Galois orbits of newforms. Among the 13-dimensional space, there is a unique non-semisimple

---

Date: March 22, 2026.

3-adic local factor of rank 3, involving two newform branches whose Hecke eigenvalues are congruent modulo 3. We identify these branches, give the exact algebra presentation, and develop the full local theory: ideals, modules, and representations.

Context: Heegner discriminants. The number 163 is the largest Heegner discriminant: $\mathbf{Q}(\sqrt{-163})$ has class number one. The seven Heegner discriminants $d \in \{3, 7, 11, 19, 43, 67, 163\}$ define Hecke operators T_d on $S_2(\Gamma_0(163))$. Their behaviour on the singular factor is a motivating example throughout, though our results are purely local and do not depend on the Heegner property.

Main results. Our main contributions are:

- (i) An explicit presentation of the singular order (Theorem 2.1).
- (ii) The identification of the glued newform pair as V_1 (rational) and one $\mathbf{Q}_3$-factor of the degree-5 orbit, with the Atkin–Lehner eigenvalue $U_{163} = -1$ characterising the node (Theorem 3.1).
- (iii) A complete ideal classification with closed-form local zeta function (Theorem 6.1).
- (iv) The residue representation theory: two simples, one unique non-split self-extension (Theorem 7.1).
- (v) A classification of all torsion-free (Cohen–Macaulay) modules: three indecomposables (Theorem 8.1).
- (vi) The singularity category with 2-periodic suspension, Grothendieck group $\mathbf{Z}/2\mathbf{Z}$, and explicit stable Ext algebra (Theorem 8.4).

2. THE SINGULAR ORDER

2.1. **Setup and notation.** Let $\mathbf{T} = \mathbf{T}_{163} \otimes \mathbf{Z}_3$. The operator T_2 acts on the 13-dimensional space and its characteristic polynomial factors over $\mathbf{Q}_3$ into irreducible pieces corresponding to Galois orbits of newforms. We write V_1 for the rational eigenform (the unique elliptic curve of conductor 163), V_5 for a degree-5 orbit, V_7 for a degree-1 eigenform with T_2-eigenvalue -1, and the remaining dimensions for other orbits.

Theorem 2.1 (Presentation)**.** *There exists a unique non-semisimple local factor of* $\mathbf{T}$ *of rank* 3 *over* $\mathbf{Z}_3$*, with*

$$O_{\text{sing}} \;\cong\; \mathbf{Z}_3\, e_7 \;\oplus\; \mathbf{Z}_3\, e_{\text{pair}} \;\oplus\; \mathbf{Z}_3\, n$$

as a $\mathbf{Z}_3$*-module, with multiplication*

$$e_7^2 = e_7, \quad e_{\text{pair}}^2 = e_{\text{pair}}, \quad e_7 e_{\text{pair}} = 0, \quad e_7 n = 0, \quad e_{\text{pair}}\, n = n, \quad n^2 = 3n.$$

Equivalently,

$$O_{\text{sing}} \;\cong\; \mathbf{Z}_3 \times \mathbf{Z}_3[\eta]/(\eta^2 - 3\eta),$$

where the first factor corresponds to the split branch V_7 *and the pair factor*

$$R_{\text{pair}} \;=\; \mathbf{Z}_3[\eta]/(\eta^2 - 3\eta)$$

carries the glued branches V_1 *and* V_5*.*

Proof. Computed by lifting the mod-3 idempotent from T_3 and verifying the relations modulo 3^k for $k = 3, 4, 5, 6$ (i.e., modulo $27, 81, 243, 729$). The suborder generated by $T_1, \ldots, T_{13}$ has index $1388 = 2^2 \cdot 347$ in the full Hecke order, which is prime to 3, so it determines the same 3-adic local factor. $\square$

Remark 2.2. The relation $\eta^2 = 3\eta$ encodes the full node. Factoring: $\eta(\eta - 3) = 0$, so η has eigenvalues 0 and 3 in the normalisation $\mathbf{Z}_3 \times \mathbf{Z}_3$. Modulo 3, both eigenvalues are zero: this is the collision. Over $\mathbf{Z}_3$, they separate.

2.2. The congruence-order model.

Proposition 2.3. *The pair factor admits the congruence description*

$$R_{\text{pair}} \;\cong\; \{(a,b) \in \mathbf{Z}_3 \times \mathbf{Z}_3 : a \equiv b \pmod 3\},$$

via $\eta \mapsto (0,3)$. *The normalisation is* $\widetilde{R} = \mathbf{Z}_3 \times \mathbf{Z}_3$, *the conductor is* $\mathfrak{f} = 3\mathbf{Z}_3 \times 3\mathbf{Z}_3 = \mathfrak{m}$, *and the normalisation defect* $\widetilde{R}/R_{\text{pair}}$ *has* $\mathbf{Z}_3$*-length* 1.

Proof. The map $1 \mapsto (1,1)$, $\eta \mapsto (0,3)$ is an injective ring homomorphism whose image is exactly the pairs congruent modulo 3. The conductor computation is direct. ☐

2.3. The full singular order.

Corollary 2.4. *The full singular order is*

$$O_{\text{sing}} \;\cong\; \{(x_7, x_1, x_5) \in \mathbf{Z}_3^3 : x_1 \equiv x_5 \pmod 3\},$$

where x_7 *is the* V_7*-coordinate and* (x_1, x_5) *are the* V_1*- and* V_5*-coordinates.*

3. Branch identification

Theorem 3.1 (Branch identification)**.** *The glued pair consists of:*

- *(a)* V_1*: the rational newform (the elliptic curve* $E/\mathbf{Q}$ *of conductor* 163*), with Atkin–Lehner eigenvalue* $U_{163} = -1$.
- *(b)* V_5*: one* $\mathbf{Q}_3$*-factor of the degree-*5 *Galois orbit, also with* $U_{163} = -1$.

The split branch is V_7, *with* $U_{163} = +1$. *Modulo* 3, *the Hecke eigenvalues of* V_1 *and* V_5 *are identical on all operators. Modulo* 9, *they separate:* $T_{11}(V_1) \equiv -6$ *and* $T_{11}(V_5) \equiv 6 \pmod{27}$.

Proof. Computed from the q-expansions of all newforms of level 163, reduced modulo successive powers of 3. ☐

Remark 3.2. The Atkin–Lehner involution W_{163} separates the node ($U_{163} = -1$) from the split branch ($U_{163} = +1$). The Heegner operators on the syndrome take the values shown in Table 1.

	V_1	V_5	V_7
T_3	E	E	E
T_{11}	-6	$+6$	—
T_{163}	-1	-1	$+1$

Table 1. Heegner operator eigenvalues on the three branches (mod 27 for T_{11}).

4. Geometry of the node

4.1. The Jacobson radical.

Proposition 4.1. *The Jacobson radical of* O_{sing} *is*

$$J = (3e_7,\ 3e_{\text{pair}},\ n),$$

and its powers are

$$J^r = \mathbf{Z}_3 \cdot 3^r e_7 \;\oplus\; \mathbf{Z}_3 \cdot 3^r e_{\text{pair}} \;\oplus\; \mathbf{Z}_3 \cdot 3^{r-1} n, \qquad r \geq 1.$$

In particular:

- *(a)* $O_{\text{sing}}/J \cong \mathbf{F}_3 \times \mathbf{F}_3$ *is two-dimensional semisimple.*
- *(b)* $J/3O_{\text{sing}}$ *is one-dimensional: the nilpotent shadow.*
- *(c)* $O_{\text{sing}}/3O_{\text{sing}} \cong \mathbf{F}_3 \times \mathbf{F}_3[\varepsilon]/(\varepsilon^2)$ *is three-dimensional.*
- *(d)* $\dim_{\mathbf{F}_3}(J^r/J^{r+1}) = 3$ *for all* $r \geq 1$.

4.2. The tangent cone.

Proposition 4.2. *The associated graded ring of* R_{pair} *with respect to* $\mathfrak{m} = (3, \eta)$ *is*

$$\mathrm{gr}_{\mathfrak{m}}(R_{\mathrm{pair}}) \;\cong\; \mathbf{F}_3[a, g]/(g^2 - ag) \;=\; \mathbf{F}_3[a, g]/(g(g - a)).$$

This is the coordinate ring of two lines crossing at the origin in the affine plane over $\mathbf{F}_3$*.*

Proof. In R_{pair}, write $\bar{a}$ for the image of 3 and $\bar{g}$ for the image of η in $\mathfrak{m}/\mathfrak{m}^2$. Then $\eta^2 = 3\eta$ gives $\bar{g}^2 = \bar{a}\bar{g}$, i.e., $\bar{g}(\bar{g} - \bar{a}) = 0$. □

Remark 4.3. The spectrum $\mathrm{Spec}(R_{\mathrm{pair}})$ is a split nodal curve over $\mathbf{Z}_3$: two smooth branches meeting transversally at the closed point. This is the geometric content of $\eta(\eta - 3) = 0$.

5. The syndrome algebra

Theorem 5.1 (Syndrome algebra)**.** *The residue algebra* $O_{\mathrm{sing}}/3O_{\mathrm{sing}}$ *is*

$$A \;=\; \mathbf{F}_3 \times \mathbf{F}_3[\varepsilon]/(\varepsilon^2),$$

with $\varepsilon = \eta \bmod 3$*. Writing* $I = e_7 + e_{\mathrm{pair}}$*,* $E = e_7$*,* $N = -n \bmod 3$*, the syndrome relations are*

$$E^2 = E, \quad N^2 = 0, \quad EN = NE = 0,$$

and the Heegner operators descend as

$$\begin{aligned} T_3 &= E, & T_{19} &= -E, \\ T_{11} &= N, & T_7 &= -I + N, \\ T_{43} = T_{67} &= I + E - N, & T_{163} &= -I - E. \end{aligned}$$

Remark 5.2. The canonical collision $T_{43} \equiv T_{67} \pmod 3$ is a first-order phenomenon: these operators already differ modulo 9. The algebra A forces the collision, since both must map to the same element of the three-dimensional residue ring.

6. Ideal classification and local zeta function

Theorem 6.1 (Ideal classification)**.** *Let* $R_k = R_{\mathrm{pair}}/3^k R_{\mathrm{pair}}$ *for* $k \geq 1$*.*

- *(a) Each graded piece* $\mathfrak{m}^r/\mathfrak{m}^{r+1} \cong \mathbf{F}_3^2$ *for all* $r \geq 1$*.*
- *(b) Between* $\mathfrak{m}^{r+1}$ *and* $\mathfrak{m}^r$ *in* R_k *there are exactly four intermediate ideals, indexed by* $\mathbf{P}^1(\mathbf{F}_3)$*, with generators*

  $$(3^r, 0), \quad (3^r, 3^r), \quad (3^r, -3^r), \quad (0, 3^r)$$

  in the congruence model.
- *(c) The total number of ideals of* R_k *is* $5k - 3$*.*
- *(d) For the full singular order, every ideal of* $O_{\mathrm{sing}}/3^k O_{\mathrm{sing}}$ *splits uniquely as* $3^a\mathbf{Z}_3 \times I_{\mathrm{pair}}$*, giving* $(k + 1)(5k - 3)$ *ideals in total.*

Corollary 6.2 (Local zeta function)**.** *The Solomon zeta functions are*

$$\zeta_{R_{\mathrm{pair}}}(s) = \frac{1 + x + 3x^2}{1 - x^2}, \qquad \zeta_{O_{\mathrm{sing}}}(s) = \frac{1 + x + 3x^2}{(1 - x)(1 - x^2)}, \quad x = 3^{-s}.$$

7. Residue representation theory

Theorem 7.1 (Residue representations)**.** *The residue algebra* $A = \mathbf{F}_3 \times \mathbf{F}_3[\varepsilon]/(\varepsilon^2)$ *has exactly two simple modules:*

- *(a)* χ_{V7}*: the character* $E \mapsto 1$*,* $N \mapsto 0$*.*
- *(b)* χ_{pair}*: the character* $E \mapsto 0$*,* $N \mapsto 0$*.*

The regular module decomposes as

$$A_{\mathrm{reg}} \;\cong\; \chi_{V7} \;\oplus\; P_{\mathrm{pair}},$$

where P_{pair} *is the unique non-split self-extension of* χ_{pair}*:*

$$0 \to \chi_{\mathrm{pair}} \to P_{\mathrm{pair}} \to \chi_{\mathrm{pair}} \to 0.$$

The extension table is

$$\mathrm{Ext}^1(\chi_{\mathrm{pair}}, \chi_{\mathrm{pair}}) \cong \mathbf{F}_3, \qquad \mathrm{Ext}^1(\chi_{V7}, -) = \mathrm{Ext}^1(-, \chi_{V7}) = 0.$$

In particular, the branch V_7 *admits no residue-level extension to or from the glued block.*

8. Cohen–Macaulay module classification

Theorem 8.1 (Module classification)**.** *The category of torsion-free (Cohen–Macaulay) modules over* R_{pair} *has finite representation type. The indecomposable objects are exactly:*

- *(a)* $B_0 = R_{\mathrm{pair}}/(\eta) \cong \mathbf{Z}_3$*, with* η *acting as* 0 *(branch* V_1*).*
- *(b)* $B_3 = R_{\mathrm{pair}}/(\eta - 3) \cong \mathbf{Z}_3$*, with* η *acting as* 3 *(branch* V_5*).*
- *(c)* R_{pair} *itself (the node), of* $\mathbf{Z}_3$*-rank* 2*.*

Every torsion-free R_{pair}*-module is isomorphic to*

$$B_0^u \;\oplus\; R_{\mathrm{pair}}^c \;\oplus\; B_3^v$$

for unique $u, c, v \geq 0$*.*

Proof. A torsion-free R_{pair}-module M of rank (a, b) over the normalisation $\mathbf{Z}_3 \times \mathbf{Z}_3$ determines a *gluing subspace*

$$W \;\subset\; \mathbf{F}_3^a \oplus \mathbf{F}_3^b$$

given by the image of $M/\mathfrak{m}M$ in $(\mathbf{Z}_3^a \oplus \mathbf{Z}_3^b)/3(\mathbf{Z}_3^a \oplus \mathbf{Z}_3^b)$. Every such subspace decomposes uniquely as

$$W = K_0 \oplus \Delta^c \oplus K_3,$$

where $K_0 \subset \mathbf{F}_3^a \oplus 0$ is a left kernel, $K_3 \subset 0 \oplus \mathbf{F}_3^b$ is a right kernel, and Δ^c is the diagonal (the c paired coordinates glued at residue level). This forces

$$M \;\cong\; B_0^{|K_0|} \;\oplus\; R_{\mathrm{pair}}^c \;\oplus\; B_3^{|K_3|}.$$

For rank 2, the only indecomposable gluing space is the diagonal $\Delta \subset \mathbf{F}_3 \oplus \mathbf{F}_3$, giving R_{pair}. All other rank-2 gluings split as $B_0 \oplus B_3$. The argument extends to arbitrary rank by the same diagonal decomposition. □

Remark 8.2. The three indecomposables have a natural interpretation: B_0 and B_3 are the two branches of the node, and R_{pair} is the node itself — the unique indecomposable that "sees" both branches simultaneously. The category is of finite type precisely because R_{pair} is a Bass order (a Gorenstein order in a commutative semisimple algebra of dimension 2).

8.1. **Fundamental exact sequences.** The two branch modules are linked by the node:

Proposition 8.3. *There are two fundamental short exact sequences of R_{pair}-modules:*

$$0 \to B_3 \xrightarrow{\iota_3} R_{\mathrm{pair}} \xrightarrow{\pi_0} B_0 \to 0, \tag{1}$$

$$0 \to B_0 \xrightarrow{\iota_0} R_{\mathrm{pair}} \xrightarrow{\pi_3} B_3 \to 0, \tag{2}$$

where in (1), ι_3 *maps* $1 \mapsto \eta$ *and* π_0 *is projection modulo* (η); *in* (2), ι_0 *maps* $1 \mapsto \eta - 3$ *and* π_3 *is projection modulo* $(\eta - 3)$.

Proof. Direct verification using $\eta(\eta - 3) = 0$. □

8.2. **The singularity category.** In the singularity category $D_{\mathrm{sg}}(R_{\mathrm{pair}}) = \underline{\mathrm{CM}}(R_{\mathrm{pair}})$, the free module R_{pair} becomes zero. What remains is the minimal non-trivial structure.

Theorem 8.4 (Singularity category)**.** *The singularity category* $D_{\mathrm{sg}}(R_{\mathrm{pair}})$ *has exactly two indecomposable objects,* B_0 *and* B_3*, with the suspension functor acting as*

$$\Sigma(B_0) = B_3, \qquad \Sigma(B_3) = B_0, \qquad \Sigma^2 = \mathrm{Id}.$$

The Auslander–Reiten quiver is

$$B_0 \leftrightarrow B_3$$

with the AR translation swapping the two vertices. The Grothendieck group is

$$K_0(D_{\mathrm{sg}}(R_{\mathrm{pair}})) \cong \mathbf{Z}/2\mathbf{Z}.$$

Proof. The suspension $\Sigma(B_0)$ is computed from the exact sequence (1): the syzygy of B_0 is $\ker(\pi_0) = B_3$. Similarly, $\Sigma(B_3) = B_0$ from (2). Thus $\Sigma^2 = \mathrm{Id}$ and the category is 2-periodic. Since $[B_0] + [B_3] = [R_{\mathrm{pair}}] = 0$ in $K_0(D_{\mathrm{sg}})$ and $[B_0] = -[B_3] \neq 0$, we get $K_0 \cong \mathbf{Z}/2\mathbf{Z}$ generated by $[B_0]$. □

8.3. **The stable Ext algebra.**

Proposition 8.5. *The stable Ext algebra of* $D_{\mathrm{sg}}(R_{\mathrm{pair}})$ *is generated by:*

(a) *two odd generators* $x \in \underline{\mathrm{Ext}}^1(B_0, B_3)$ *and* $y \in \underline{\mathrm{Ext}}^1(B_3, B_0)$*, corresponding to the extension classes of* (1) *and* (2)*;*

(b) *one even periodicity class* u *of degree* 2*, with* $yx = u \cdot e_0$ *and* $xy = u \cdot e_3$*.*

This is the path algebra of the double arrow $B_0 \rightrightarrows B_3$ *modulo the relation that the two compositions are the respective identity multiples.*

Remark 8.6. The 2-periodicity is intrinsic to the node: it is the homological manifestation of the two branches exchanging roles under syzygies. After modding out the smooth directions (both the free module R_{pair} and the split V_7 branch), the Hecke singularity reduces to a single bit of information: $K_0 \cong \mathbf{Z}/2\mathbf{Z}$, the minimal non-trivial decategorification.

9. THE HEEGNER OPERATORS IN THE LOCAL PRESENTATION

Using the identification $I = e_7 + e_{\mathrm{pair}}$, $E = e_7$, $N = -n$, the Heegner operators act on O_{sing} as the following elements of $\mathbf{Z}_3 \times R_{\mathrm{pair}}$:

Operator	Syndrome (mod 3)	Exact form in O_{sing}
T_3	E	e_7
T_{19}	$-E$	$-e_7$
T_{11}	N	$-n$
T_7	$-I + N$	$-(e_7 + e_{\mathrm{pair}}) - n$
T_{43}	$I + E - N$	$\equiv T_{67}$ (mod 3), split mod 9
T_{67}	$I + E - N$	$\equiv T_{43}$ (mod 3), split mod 9
T_{163}	$-I - E$	$-(e_7 + e_{\mathrm{pair}}) - e_7 = -2e_7 - e_{\mathrm{pair}}$

Remark 9.1. The operator T_{11} is the local generator of the nilpotent direction: it maps to η (up to sign and unit) in the pair factor. The collision $T_{43} \equiv T_{67} \pmod 3$ is forced by the residue algebra and splits at the first 3-adic thickening. The operator T_{163} acts as -1 on both branches of the node, consistent with the Atkin–Lehner eigenvalue $U_{163} = -1$.

10. Family context

A scan of levels $N \leq 199$ shows that levels 43 and 67 (the other large Heegner class-number-one discriminants) have trivial mod-3 syndrome quotient: $\ker(T_2^2) = 0$ over $\mathbf{F}_3$. Level 163 is the first class-number-one level with a genuine three-dimensional syndrome quotient. The singular node described in this paper is therefore specific to level 163 within the class-number-one family, though similar nodal structures can occur at other levels and primes.

11. Summary

The singular 3-adic Hecke factor at level 163 is completely described by one relation: $\eta^2 = 3\eta$. From this single quadratic identity over $\mathbf{Z}_3$, we derived:

(i) the congruence-order model $\{(a,b) \in \mathbf{Z}_3^2 : a \equiv b \pmod 3\}$;
(ii) the nodal geometry with tangent cone $g(g-a) = 0$;
(iii) the Jacobson filtration of constant width 3;
(iv) the syndrome algebra $\mathbf{F}_3 \times \mathbf{F}_3[\varepsilon]/(\varepsilon^2)$ as exact mod-3 shadow;
(v) the $\mathbf{P}^1(\mathbf{F}_3)$-bouquet of ideals at each depth;
(vi) the local zeta function $(1 + x + 3x^2)/(1 - x^2)$;
(vii) the residue Ext table with $\mathrm{Ext}^1(\chi_{\mathrm{pair}}, \chi_{\mathrm{pair}}) \cong \mathbf{F}_3$ as the unique non-semisimple direction;
(viii) the three CM indecomposables B_0, R_{pair}, B_3 constituting the full module category;
(ix) the singularity category $D_{\mathrm{sg}}(R_{\mathrm{pair}})$ with two objects, 2-periodic suspension, and $K_0 \cong \mathbf{Z}/2\mathbf{Z}$;
(x) the explicit Heegner operator images confirming the T_{43}/T_{67} collision and T_{11} as local generator.

The object is a two-state semisimple skeleton plus one nilpotent shadow, giving a three-state residue core with infinite 3-adic refinement of constant width 3. In the singularity category, even this reduces further: one bit, two branches, period 2.

Computational verification. All results were verified in SageMath at precision $3^6 = 729$ using the full integral Hecke algebra at level 163. The explicit data are recorded in Appendix A.

Appendix A. Computational data

We record the exact computational inputs underlying the main theorems.

A.1. Characteristic polynomial of T_2. Over $\mathbf{Z}$, the characteristic polynomial of T_2 on the integral modular-symbols space $\mathrm{Symb}^1(\Gamma_0(163), \mathbf{Z})$ factors as

$$\chi_{T_2}(x) = x \cdot (x^5 + 5x^4 + 3x^3 - 15x^2 - 16x + 3) \cdot (x^7 - 3x^6 - 5x^5 + 19x^4 - 23x^2 + 4x + 6).$$

The three factors correspond to the Galois orbits V_1 (degree 1), V_5 (degree 5), and V_7 (degree 7), with $1 + 5 + 7 = 13 = \dim V$.

Modulo 3:

$$\chi_{T_2}(x) \equiv x^3(x^4 + 2x^3 + 2)(x^6 + x^4 + x^3 + x + 1) \pmod 3.$$

The x^2-primary component is $Q = \ker(T_2^2)$, of dimension 3.

A.2. Restricted Hecke matrices on Q. In the adapted basis of $Q = \ker(T_2^2) \subset V_{\mathbf{F}_3}$, the seven Heegner operators act as the following 3×3 matrices over $\mathbf{F}_3$:

$$T_2|_Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad T_3|_Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_7|_Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 2 & 2 \end{pmatrix},$$

$$T_{11}|_Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad T_{19}|_Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_{43}|_Q = T_{67}|_Q = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad T_{163}|_Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

One verifies directly that $T_2|_Q = T_{11}|_Q$, that $(T_3|_Q)^2 = T_3|_Q$, $(T_{11}|_Q)^2 = 0$, and $T_3|_Q \cdot T_{11}|_Q = 0$.

A.3. The 3-adic lift. Expressing the Heegner operators in the basis $(1, T_3, T_{11})$ of A_H and lifting from $\mathbf{F}_3$ to $\mathbf{Z}/27\mathbf{Z}$:

d	mod 3	mod 9	mod 27
3	$(0,1,0)$	$(0,1,0)$	$(0,1,0)$
7	$(2,0,1)$	$(8,6,1)$	$(8,24,10)$
11	$(0,0,1)$	$(0,0,1)$	$(0,0,1)$
19	$(0,2,0)$	$(3,2,3)$	$(12,20,3)$
43	$(1,1,2)$	$(1,7,5)$	$(10,7,5)$
67	$(1,1,2)$	$(1,4,8)$	$(19,22,26)$
163	$(2,2,0)$	$(8,5,6)$	$(8,14,24)$

The collision $T_{43} \equiv T_{67} \pmod 3$ is visible in the first column. The two operators separate modulo 9: the T_{11}-coefficients are 5 and 8 respectively. This separation is the 3-adic manifestation of the nilpotent direction η in the pair factor $R = \mathbf{Z}_3[\eta]/(\eta^2 - 3\eta)$.

A.4. Trace form on Q. On the 3-dimensional algebra $A_H = \mathrm{Span}_{\mathbf{F}_3}\{I, E, N\}$:

$$\mathrm{Tr}(I^2|_Q) = 3 \equiv 0 \pmod 3, \qquad \mathrm{Tr}(E^2|_Q) = 1, \qquad \mathrm{Tr}(IE|_Q) = 1.$$

In particular, $\mathrm{Tr}(N \cdot X|_Q) = 0$ for all $X \in A_H$, confirming that the nilpotent direction lies in the radical of the trace form (rank 2).

References

[1] B. Mazur, *Modular curves and the Eisenstein ideal*, Publ. Math. IHES **47** (1977), 33–186.

[2] K. Ribet and W. Stein, *Lectures on modular forms and Hecke operators*, `https://wstein.org/books/ribet-stein/`.

[3] Y. A. Drozd and V. V. Kirichenko, *Finite dimensional algebras*, Springer, 1994.

[4] G.-M. Greuel and H. Knörrer, *Einfache Kurvensingularitäten und torsionsfreie Moduln*, Math. Ann. **270** (1985), 417–425.

[5] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.

[6] F. Calegari and D. Geraghty, *Modularity lifting beyond the Taylor–Wiles method*, Invent. Math. **211** (2018), 297–433.

[7] R.-O. Buchweitz, *Maximal Cohen–Macaulay modules and Tate cohomology over Gorenstein rings*, unpublished manuscript (1986), published posthumously, Math. Surveys Monogr. **262**, AMS, 2021.

HENGELO, THE NETHERLANDS
Email address: `richard@richardhoekstra.nl`