THE HEEGNER SYNDROME ALGEBRA ON THE CANONICAL MOD-3 HECKE QUOTIENT AT LEVEL 163

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Abstract. Let $V_{\mathbf{F}_3}$ be the mod-3 reduction of the cuspidal modular-symbols space for $\Gamma_0(163)$, and let

$$Q = \ker(T_2^2 : V_{\mathbf{F}_3} \to V_{\mathbf{F}_3}).$$

Then $\dim_{\mathbf{F}_3} Q = 3$, and Q is stable under the Hecke algebra modulo 3. We show computationally that the seven Heegner operators T_d for

$$d \in \{3, 7, 11, 19, 43, 67, 163\}$$

act on Q through a 3-dimensional commutative algebra with basis $\{I, E, N\}$ satisfying

$$E^2 = E, \qquad N^2 = EN = NE = 0.$$

Hence this algebra is isomorphic to

$$\mathbf{F}_3 \times \mathbf{F}_3[\varepsilon]/(\varepsilon^2).$$

Its projectivized unit group is cyclic of order 6, and the invertible Heegner operators generate the full projective unit group. We also compare prime levels up to 199 and show that level 163 is the first class-number-one prime level whose canonical syndrome summand is genuinely 3-dimensional.

1. Introduction

At level $N = 163$, the cuspidal modular-symbols space

$$V = S_2(\Gamma_0(163))$$

has dimension 13 over $\mathbf{Q}$. The integral modular-symbols lattice $V_{\mathbf{Z}} \subset V$ is preserved by all Hecke operators. Reducing modulo 3 gives a 13-dimensional $\mathbf{F}_3$-vector space $V_{\mathbf{F}_3} = V_{\mathbf{Z}}/3V_{\mathbf{Z}}$ with commuting Hecke operators.

Earlier computations in this project established two separate mod-3 phenomena:

(1) a canonical Hecke-stable decomposition

$$V_{\mathbf{F}_3} = \ker(T_2^2) \oplus \operatorname{im}(T_2^2)$$

of dimensions $3 + 10$;

(2) a noncanonical discrete-log construction producing a weight-5 codeword in the ternary Hamming code.

The second construction depends on a choice of primitive root modulo 163 and therefore does not give a canonical bridge between the Hecke module and the Hamming code. The purpose of this note is to record the strongest canonical bridge presently known: the action of the Heegner operators on the canonical 3-dimensional summand.

2. The canonical Hecke-stable summand

Proposition 1. *Let $V_{\mathbf{F}_3}$ be the mod-3 reduction of the cuspidal modular-symbols space at level 163. Then*

$$\dim \ker(T_2^2) = 3, \qquad \dim \operatorname{im}(T_2^2) = 10,$$

and

$$V_{\mathbf{F}_3} = \ker(T_2^2) \oplus \operatorname{im}(T_2^2).$$

Since every Hecke operator commutes with T_2, both summands are Hecke-stable.

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Proof. Modulo 3, the characteristic polynomial of T_2 factors as

$$\chi_{T_2}(x) = x^3(x^4 + 2x^3 + 2)(x^6 + x^4 + x^3 + x + 1).$$

The three factors are pairwise coprime in $\mathbf{F}_3[x]$, so primary decomposition gives

$$V_{\mathbf{F}_3} = V_{x^2} \oplus V_{x^4+2x^3+2} \oplus V_{x^6+x^4+x^3+x+1},$$

with dimensions 3, 4, and 6 respectively. Since $\ker(T_2^2) = V_{x^2}$ has dimension 3 and $\operatorname{im}(T_2^2)$ is the complementary summand of dimension 10, the result follows. □

Definition 2. Set

$$Q := \ker(T_2^2) \subset V_{\mathbf{F}_3}.$$

Then Q is a 3-dimensional Hecke-stable direct summand of $V_{\mathbf{F}_3}$.

Remark 3. *The summand Q is canonical in the following sense: it is the x^2-primary component of T_2 acting on $V_{\mathbf{F}_3}$. The choice of T_2 as the splitting operator is natural since 2 is the smallest prime not dividing the level. However, the same 3-dimensional summand arises as a Hecke-stable piece for any operator whose mod-3 reduction separates the three primary components.*

3. The Heegner syndrome algebra

Let

$$\mathcal{H} = \{3, 7, 11, 19, 43, 67, 163\}$$

be the Heegner operator set at level 163.

Theorem 4. *The restricted Heegner operators*

$$T_d|_Q \in \operatorname{End}_{\mathbf{F}_3}(Q), \qquad d \in \mathcal{H},$$

span a 3-dimensional commutative algebra $A_Q \subset \operatorname{End}(Q)$.

More precisely, with

$$I = \operatorname{Id}_Q, \qquad E = T_3|_Q, \qquad N = T_{11}|_Q,$$

one has

$$E^2 = E, \qquad N^2 = 0, \qquad EN = NE = 0,$$

and the Heegner operators satisfy

$$\begin{aligned} T_3|_Q &= E, \\ T_{19}|_Q &= -E, \\ T_{11}|_Q &= N, \\ T_7|_Q &= -I + N, \\ T_{43}|_Q = T_{67}|_Q &= I + E - N, \\ T_{163}|_Q &= -I - E. \end{aligned}$$

In particular,

$$A_Q = \operatorname{Span}_{\mathbf{F}_3}\{I, E, N\},$$

and A_Q is generated by one idempotent and one orthogonal nilpotent.

Proof. The matrices were computed explicitly on Q in an adapted modular-symbols basis. Direct multiplication gives the stated relations. Since all seven Heegner operators are linear combinations of I, E, N, the span has dimension at most 3. Since I, E, N are linearly independent, the dimension is exactly 3. □

Corollary 5. *The syndrome algebra is isomorphic to*

$$A_Q \cong \mathbf{F}_3 \times \mathbf{F}_3[\varepsilon]/(\varepsilon^2).$$

Proof. Put $F = I - E$. Then

$$E^2 = E, \qquad F^2 = F, \qquad EF = FE = 0, \qquad FN = NF = N.$$

Thus E spans a copy of $\mathbf{F}_3$, while (F, N) spans the dual-number algebra $\mathbf{F}_3[\varepsilon]/(\varepsilon^2)$ with ε corresponding to N. □

Remark 6. *The canonical collision*

$$T_{43}|_Q = T_{67}|_Q$$

is exact. This is the strongest base-free identification between Heegner labels currently visible on the mod-3 side.

4. The quotient trace form

Proposition 7. *Define the quotient trace form on $\mathcal{H}$ by*

$$G_Q(d, d') = \mathrm{Tr}(T_d T_{d'}|_Q).$$

Then G_Q has rank 2.

Proof. In the commutative algebra A_Q, multiplication by the nilpotent element N is a nilpotent endomorphism: its only eigenvalue is 0, so $(NX) = 0$ for all $X \in A_Q$. Hence the N-line lies in the radical of the trace form, and the trace form on A_Q has rank at most 2. The explicit values $(I^2|_Q) = 3 \equiv 0$, $(E^2|_Q) = 1$, $(IE|_Q) = 1$ show the rank is exactly 2 over $\mathbf{F}_3$. □

Remark 8. *Thus the quotient retains a 3-dimensional Hecke algebra but only a 2-dimensional trace geometry. The nilpotent direction survives algebraically but is invisible to trace.*

5. The projective unit group

Theorem 9. *The unit group of A_Q has 12 elements, and its projectivization modulo scalar units is cyclic of order 6:*

$$A_Q^\times/\mathbf{F}_3^\times \cong C_6.$$

The invertible Heegner operators $T_7|_Q$, $T_{43}|_Q$, $T_{67}|_Q$, and $T_{163}|_Q$ generate the full projective unit group.

Proof. From Corollary 5,

$$A_Q^\times \cong \mathbf{F}_3^\times \times \left(\mathbf{F}_3[\varepsilon]/(\varepsilon^2)\right)^\times.$$

Now $\mathbf{F}_3^\times$ has order 2, while

$$\left(\mathbf{F}_3[\varepsilon]/(\varepsilon^2)\right)^\times \cong \mathbf{F}_3^\times \times (1 + \varepsilon\mathbf{F}_3) \cong C_2 \times C_3 \cong C_6.$$

Hence $|A_Q^\times| = 2 \cdot 6 = 12$, and modding out by scalar units gives a group of order 6, necessarily isomorphic to C_6.

The explicit projective permutations induced by the invertible Heegner operators have orders

$$|T_7| = 3, \qquad |T_{43}| = |T_{67}| = 6, \qquad |T_{163}| = 2,$$

and the subgroup they generate has order 6. Therefore it is the full projective unit group. □

Corollary 10. *The projective action of $A_Q^\times/\mathbf{F}_3^\times$ on the 13 points of $\mathbf{P}(Q)$ has orbit sizes*

$$1 + 1 + 2 + 3 + 6.$$

Proof. Direct computation of the projective unit action on $\mathbf{P}(Q)$. □

Remark 11. *This explains the previously observed order-6 projective Hecke group exactly: it is the full projectivized unit group of the syndrome algebra, not an accidental subgroup of* $\mathrm{PGL}(3,3)$.

6. Family context

Proposition 12. *Among the class-number-one prime levels*

$$19,\ 43,\ 67,\ 163,$$

*the dimensions of the canonical mod-*3 *quotient* $Q_N = \ker(T_2^2)$ *are*

$$1,\ 0,\ 0,\ 3$$

respectively. Thus 163 *is the first class-number-one level with a genuinely* 3*-dimensional syndrome quotient.*

Proof. Direct computation in SageMath. □

Remark 13. *The phenomenon is not globally unique: other prime levels up to* 199 *also have nontrivial* Q_N*. What is special within the class-number-one family is the first appearance of the full* 3*-dimensional syndrome space.*

7. Discussion

The canonical object on the mod-3 side is not a seven-point configuration in $\mathbf{P}^2(\mathbf{F}_3)$. The seven Heegner primes do not canonically embed there. What is canonical is stronger in a different direction:

(1) the Hecke module has a canonical $10 + 3$ split modulo 3;
(2) the Heegner operators descend to a 3-dimensional algebra on the quotient;
(3) that algebra has an explicit structure

$$\mathbf{F}_3 \times \mathbf{F}_3[\varepsilon]/(\varepsilon^2);$$

(4) its projective unit group is exactly the order-6 Hecke symmetry seen in earlier computations, with orbit decomposition $1 + 1 + 2 + 3 + 6$ on $\mathbf{P}(Q)$.

So the Hamming–Hecke bridge that survives canonically is not a point-set bridge but an algebra-and-group bridge.

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References

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