

A TRACE INEQUALITY FOR HECKE OPERATORS AT CLASS-NUMBER-ONE HEEGNER PRIMES OF LEVEL 163

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ABSTRACT. Let

$$S = \{3, 7, 11, 19, 43, 67, 163\}$$

and, for $d, d' \in S$, define

$$G(d, d') = \text{Tr}(T_d T_{d'} \mid S_2(\Gamma_0(163))),$$

where T_{163} denotes the Atkin–Lehner operator U_{163} . We prove the inequality

$$|G(d, d')| \leq 2 \min(G(d, d), G(d', d'))$$

for all $d, d' \in S$. The proof is case-by-case: nine pairs follow from Cauchy–Schwarz, five vacuum-row pairs from the prime-level Eichler–Selberg formula, four pairs from the $1 + 5 + 7$ newform-orbit decomposition, and three residual pairs from exact finite Hurwitz-class-number identities. We also record computational enlargement data showing that the phenomenon is specific to the Heegner set and fails for larger prime sets at the same level.

1. INTRODUCTION

Let

$$V = S_2(\Gamma_0(163)).$$

Since $163 \equiv 7 \pmod{12}$, the modular curve $X_0(163)$ has genus

$$g = \dim V = 13.$$

The prime 163 is the largest class-number-one Heegner prime [1, 3, 6]. At level 163 the relevant Heegner operator set is

$$S = \{3, 7, 11, 19, 43, 67, 163\},$$

where for $d = 163$ we set $T_{163} := U_{163}$, the Atkin–Lehner operator at the level. At prime level N , the new subspace equals the full cuspidal space, and U_N acts as $-w_N$ where w_N is the Atkin–Lehner involution [2, §5.7]. This convention is used consistently throughout.

For $d, d' \in S$ define

$$G(d, d') = \text{Tr}(T_d T_{d'} \mid V).$$

The matrix $G = (G(d, d'))_{d, d' \in S}$ is symmetric (since the Hecke operators are self-adjoint with respect to the Petersson inner product and trace is cyclic) and integer-valued (since the operators preserve the integral modular-symbols lattice). Writing f_1, \dots, f_{13} for the normalized Hecke eigenforms,

$$G(d, d') = \sum_{i=1}^{13} a_d(f_i) a_{d'}(f_i),$$

so G is a Gram matrix and in particular positive semidefinite; positive definiteness follows from the computed eigenvalues below.

Our main result is the following.

Theorem 1. *For all $d, d' \in S$ one has*

$$|\text{Tr}(T_d T_{d'} \mid V)| \leq 2 \min(\text{Tr}(T_d^2 \mid V), \text{Tr}(T_{d'}^2 \mid V)).$$

The proof is computationally exact but not uniform. It splits the 21 unordered pairs in S into four categories:

- (1) 9 pairs by Cauchy–Schwarz, because the larger diagonal is at most four times the smaller;

- (2) 5 vacuum-row pairs $(163, d)$ by the prime-level Eichler–Selberg formula;
- (3) 4 pairs by orbitwise Cauchy–Schwarz on the $1 + 5 + 7$ newform decomposition;
- (4) 3 residual pairs by explicit finite Hurwitz-class-number sums.

Remark 2. The proof is therefore exact but case-by-case. We do not know a uniform conceptual proof of Theorem 1.

2. THE HECKE TRACE GRAM MATRIX

Proposition 3. *With rows and columns ordered as $(3, 7, 11, 19, 43, 67, 163)$, the trace matrix is*

$$G = \begin{pmatrix} 46 & -18 & -12 & 20 & -44 & -70 & 6 \\ -18 & 92 & -10 & -44 & -4 & -60 & 4 \\ -12 & -10 & 136 & -38 & -88 & 120 & 6 \\ 20 & -44 & -38 & 282 & -102 & 6 & -6 \\ -44 & -4 & -88 & -102 & 583 & -164 & -13 \\ -70 & -60 & 120 & 6 & -164 & 640 & -14 \\ 6 & 4 & 6 & -6 & -13 & -14 & 13 \end{pmatrix}.$$

Its eigenvalues are approximately

$$10.37, 19.44, 76.78, 98.09, 274.14, 491.62, 821.55.$$

In particular, G is positive definite.

Proof. The entries of G are computed in SageMath [5] using the integral modular-symbols realization of V . Symmetry follows from trace cyclicity: $\mathrm{Tr}(T_d T_{d'}) = \mathrm{Tr}(T_{d'} T_d)$. The eigenvalues are computed numerically and are all positive, confirming positive definiteness. \square

Proposition 4. *One has*

$$G(163, 163) = 13 = g.$$

Proof. At prime level N , the operator U_N acts on the new subspace by $U_N = -w_N$, where w_N is the Atkin–Lehner involution [2, §5.7]. Hence

$$U_{163}^2 = w_{163}^2 = \mathrm{Id}_V,$$

so

$$G(163, 163) = \mathrm{Tr}(U_{163}^2 | V) = \dim V = 13.$$

\square

3. PROOF OF THE TRACE INEQUALITY

We write

$$B(d, d') = 2 \min(G(d, d), G(d', d')).$$

Theorem 1 is the statement that $|G(d, d')| \leq B(d, d')$ for all $d, d' \in S$.

3.1. Nine pairs by Cauchy–Schwarz.

Lemma 5. *The inequality of Theorem 1 holds for the nine pairs*

$$(3, 7), (3, 11), (3, 163), (7, 11), (7, 19), (11, 19), (19, 43), (19, 67), (43, 67).$$

Proof. Since G is a Gram matrix, Cauchy–Schwarz gives

$$|G(d, d')| \leq \sqrt{G(d, d)G(d', d')}.$$

If $\max(G(d, d), G(d', d')) \leq 4 \min(G(d, d), G(d', d'))$, then

$$\sqrt{G(d, d)G(d', d')} \leq 2 \min(G(d, d), G(d', d')).$$

The nine pairs above satisfy this ratio condition. Their diagonal ratios are listed in Table 1. \square

pair	diagonals	larger/smaller	bound
(3, 7)	(46, 92)	2.000	92
(3, 11)	(46, 136)	2.957	92
(3, 163)	(46, 13)	3.538	26
(7, 11)	(92, 136)	1.478	184
(7, 19)	(92, 282)	3.065	184
(11, 19)	(136, 282)	2.074	272
(19, 43)	(282, 583)	2.067	564
(19, 67)	(282, 640)	2.270	564
(43, 67)	(583, 640)	1.098	1166

TABLE 1. The nine pairs covered by Cauchy–Schwarz.

3.2. Five vacuum-row pairs by Eichler–Selberg. For the vacuum row we use the prime-level Eichler–Selberg trace formula specialized to weight 2 and level 163 [4, Thm. 4.2.24]. For nonsquare n prime to 163 it takes the form

$$(1) \quad \mathrm{Tr}(T_n | V) = \frac{1}{2} \sum_{t^2 < 4n} \left(1 + \left(\frac{\Delta}{163}\right)\right) H(\Delta) - \sum_{\substack{a|n \\ (a, 163)=1}} a, \quad \Delta = 4n - t^2,$$

where $H(\Delta)$ denotes the Hurwitz class number.

Lemma 6. *The inequality of Theorem 1 holds for the five pairs*

$$(7, 163), (11, 163), (19, 163), (43, 163), (67, 163).$$

Proof. Write $n = 163p$ with $p \in \{7, 11, 19, 43, 67\}$. Then

$$\Delta = 4 \cdot 163p - t^2 \equiv -t^2 \pmod{163}.$$

Since $163 \equiv 3 \pmod{4}$, one has $\left(\frac{-1}{163}\right) = -1$. Hence if $163 \nmid t$, then

$$\left(\frac{\Delta}{163}\right) = \left(\frac{-t^2}{163}\right) = -1,$$

so the factor $1 + \left(\frac{\Delta}{163}\right)$ vanishes. Because

$$t^2 < 4 \cdot 163p < (2 \cdot 163)^2,$$

the only possible surviving values are $t = 0$ and, for $p \in \{43, 67\}$, also $t = 163$.

The five exact evaluations are:

$$\mathrm{Tr}(T_{163 \cdot 7}) = \frac{1}{2}(24) - 8 = 4,$$

$$\mathrm{Tr}(T_{163 \cdot 11}) = \frac{1}{2}(36) - 12 = 6,$$

$$\mathrm{Tr}(T_{163 \cdot 19}) = \frac{1}{2}(28) - 20 = -6,$$

$$\mathrm{Tr}(T_{163 \cdot 43}) = \frac{1}{2}(52 + 10) - 44 = -13,$$

$$\mathrm{Tr}(T_{163 \cdot 67}) = \frac{1}{2}(44 + 64) - 68 = -14.$$

Since $G(163, 163) = 13$, the required bound is always $2 \cdot 13 = 26$, and each of the five absolute values is at most 26. \square

3.3. Four pairs by the 1+5+7 orbit decomposition. The space V decomposes over \mathbf{Q} into three Hecke-stable subspaces corresponding to the three Galois conjugacy classes of normalized newforms:

- V_1 : the rational eigenform, corresponding to the elliptic curve 163a1 of conductor 163 (dimension 1);
- V_5 : a conjugacy class of degree 5 over \mathbf{Q} (dimension 5);
- V_7 : a conjugacy class of degree 7 over \mathbf{Q} (dimension 7).

This gives $1 + 5 + 7 = 13 = \dim V$. Accordingly, the trace Gram matrix splits as

$$G = C_1 + C_5 + C_7,$$

where

$$C_k(d, d') = \operatorname{Tr}(T_d T_{d'} | V_k) = \sum_{f \in V_k} a_d(f) a_{d'}(f).$$

At level 163 the Atkin–Lehner signs are

$$w_{163} = (+1, +1, -1)$$

on (V_1, V_5, V_7) , equivalently

$$U_{163} = (-1, -1, +1).$$

Lemma 7. *The inequality of Theorem 1 holds for the four pairs*

$$(3, 19), (7, 43), (11, 43), (11, 67).$$

Proof. For each orbit one has orbitwise Cauchy–Schwarz

$$|C_k(d, d')| \leq \sqrt{C_k(d, d) C_k(d', d')}.$$

Using the exact orbit decompositions from the modular-symbol computation gives the bounds in Table 2. In each case the resulting orbitwise bound is below the required bound $B(d, d')$. \square

pair	exact $C_1 + C_5 + C_7$	orbitwise bound	required bound $B(d, d')$
(3, 19)	$0 + 0 + 20$	50.07	92
(7, 43)	$14 + 10 - 28$	46.40	184
(11, 43)	$-42 + 1 - 47$	257.85	272
(11, 67)	$12 + 96 + 12$	253.80	272

TABLE 2. The four pairs proved from the $1 + 5 + 7$ orbit decomposition.

3.4. Three residual pairs by exact finite sums.

Lemma 8. *The inequality of Theorem 1 holds for the three residual pairs*

$$(3, 43), (3, 67), (7, 67).$$

Proof. These are the indices $n = 129 = 3 \cdot 43$, $n = 201 = 3 \cdot 67$, and $n = 469 = 7 \cdot 67$. The divisor sums are

$$\sum_{(a,163)=1} a|_{129} a = 176, \quad \sum_{(a,163)=1} a|_{201} a = 272, \quad \sum_{(a,163)=1} a|_{469} a = 544.$$

For each n , the Eichler–Selberg class-number sum runs over those integers t with $t^2 < 4n$ for which $\left(\frac{4n-t^2}{163}\right) \neq -1$. The individual terms $\left(1 + \left(\frac{\Delta}{163}\right)\right) H(\Delta)$ for the surviving t -values are listed in Table 3. The exact evaluations are:

$$\begin{aligned} \operatorname{Tr}(T_{129}) &= \frac{1}{2}(24 + 32 + 72 + 32 + 16 + 20 + 32 + 20 + 16) - 176 = -44, \\ \operatorname{Tr}(T_{201}) &= \frac{1}{2}(24 + 40 + 16 + 40 + 64 + 16 + 40 + 72 + 32 + 24 + 16 + 20) - 272 = -70, \\ \operatorname{Tr}(T_{469}) &= \frac{1}{2}(32 + 20 + 56 + 80 + 48 + 128 + 40 + 32 + 48 + 96 \\ &\quad + 16 + 96 + 24 + 24 + 40 + 72 + 32 + 20 + 32 + 32) - 544 = -60. \end{aligned}$$

These satisfy

$$|-44| < 92, \quad |-70| < 92, \quad |-60| < 184,$$

which are exactly the required bounds $B(3, 43) = 92$, $B(3, 67) = 92$, $B(7, 67) = 184$. \square

Proof of Theorem 1. Combine Lemmas 5, 6, 7, and 8. \square

n	surviving $ t $ -values	$\frac{1}{2} \sum (1 + (\frac{\Delta}{163})) H(\Delta)$
129	1, 8, 10, 11, 12, 13, 16, 17, 19	132
201	0, 1, 3, 4, 8, 9, 13, 14, 15, 17, 24, 25	202
469	0, 3, 5, 6, 7, 10, 13, 15, 18, 20, 21, 22, 23, 27, 29, 30, 31, 33, 36, 40	544

TABLE 3. Surviving Eichler–Selberg terms for the three residual indices. For each n , the surviving t -values are those with $t^2 < 4n$ and $(\frac{4n-t^2}{163}) \neq -1$. Each $|t| > 0$ contributes twice (from $\pm t$). The column headers show $|t|$ only; the half-sum accounts for multiplicities.

4. ENLARGEMENT DATA

The trace inequality is specific to the Heegner set S and does not persist under arbitrary enlargement.

Proposition 9. *At level 163:*

- (1) *adjoining the individual primes 17, 53, 71, 89, 97 to S already produces violations of the trace inequality;*
- (2) *adjoining all primes ≤ 97 together with 163 produces eight violations, the worst being*

$$|G(53, 163)| = 43 > 26.$$

Proof. Direct computation gives the following individual failures:

$$|G(17, 163)| = 34 > 26,$$

$$|G(53, 163)| = 43 > 26,$$

$$|G(71, 3)| = 100 > 92, \quad |G(71, 163)| = 29 > 26,$$

$$|G(89, 163)| = 32 > 26,$$

$$|G(97, 3)| = 106 > 92.$$

For the full enlarged set of primes ≤ 97 together with 163, the computational scan records eight violations, with worst pair (53, 163) of ratio $43/26 \approx 1.654$. \square

Remark 10. The saved scan data do not support the stronger statement that S is inclusion-maximal among all subsets of primes ≤ 97 . For example, the one-prime extensions by

$$2, 5, 13, 23, 29, 31, 37, 41, 47, 59, 61, 73, 79, 83$$

still satisfy the trace inequality.

5. REMARKS

Remark 11. By Proposition 4,

$$163 - 2G(163, 163) = 163 - 26 = 137.$$

We record this as an arithmetic coincidence:

$$163 - 2g = 137.$$

Remark 12. The inequality of Theorem 1 is not generic for Hecke operators. For example, at level 163 one has

$$|G(53, 163)| = 43 > 26 = 2 \min(G(53, 53), G(163, 163)),$$

so the Heegner-set phenomenon fails even at the same level once the operator set is enlarged.

Remark 13. The proof given here is exact, but it does not explain the inequality conceptually. The vacuum row is controlled by a local factor in the prime-level trace formula, whereas the off-vacuum residual cases still require explicit whole-sum cancellation. A uniform proof remains open.

Acknowledgements. Computations were carried out in SageMath, NumPy, and SciPy.

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